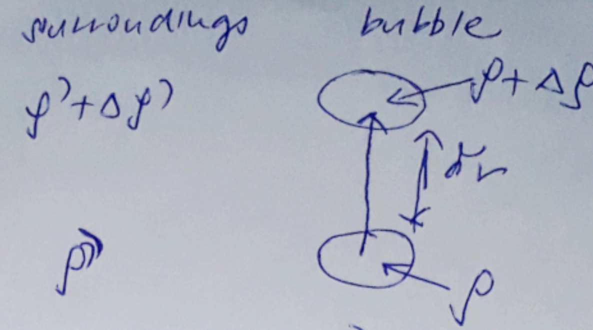
Convection

Mixing - length theory

- energy transport by convection



$\rho' > \rho \rightarrow$ rise...

equation of motion for the bubble:

$$\rho \frac{d^2 \delta r}{dt^2} = -g \Delta\rho = -g \left[\left(\frac{d\rho}{dr}\right)_{ad} - \left(\frac{d\rho}{dr}\right)_{rad} \right] \delta r$$

$$\rho = \frac{\mu P}{RT} \; ; \; \frac{d\mu}{dr} \overset{!}{=} 0 \Rightarrow \frac{d\rho}{dr} = \frac{\mu}{RT}\frac{dP}{dr} - \frac{\mu P}{R}\frac{1}{T^2}\frac{dT}{dr}$$

$$\left(\frac{d\rho}{dr}\right)_{ad} - \left(\frac{d\rho}{dr}\right)_{rad} = \cancel{\frac{\mu}{RT}\frac{dP}{dr}} - \left(\frac{\mu P}{RT^2}\frac{dT}{dr}\right)_{ad} -$$

$$- \left[\cancel{\frac{\mu}{RT}\frac{dP}{dr}} - \left(\frac{\mu P}{RT^2}\frac{dT}{dr}\right)_{rad} \right] =$$

$$= -\frac{\mu P}{RT^2} \left[\left(\frac{dT}{dr}\right)_{ad} - \left(\frac{dT}{dr}\right)_{rad} \right] =$$

$$= \frac{\mu P}{RT^2} \left[\left(\frac{dT}{d\ln P}\right)_{ad} \frac{d\ln P}{dr} - \left(\frac{dT}{d\ln P}\right)_{rad} \frac{d\ln P}{dr} \right] =$$

$$= -\frac{\mu P}{RT^2} \underbrace{\left(\frac{d\ln P}{dr}\right)}_{-\frac{1}{H_p}} T \left[\left(\frac{d\ln T}{d\ln P}\right)_{ad} - \left(\frac{d\ln T}{d\ln P}\right)_{rad} \right] =$$

$$= + \overset{\rho}{\left(\frac{\mu P}{RT}\right)} \frac{1}{H_p} \left[\left(\frac{d\ln T}{d\ln P}\right)_{ad} - \left(\frac{d\ln T}{d\ln P}\right)_{rad} \right]$$

$$\rho \frac{d^2 \delta r}{dt^2} = -g \rho \frac{1}{H_p} \left[\underbrace{\left(\frac{d\ln T}{d\ln P}\right)_{ad}}_{\nabla_{ad}} - \underbrace{\left(\frac{d\ln T}{d\ln P}\right)_{rad}}_{\nabla} \right] \delta r$$

$$\frac{d^2 \delta r}{dt^2} = -\frac{g}{H_p} \left(\nabla_{ad} - \nabla \right) \delta r = -N^2 \delta r$$

$$N^2 = \frac{g}{H_p} \left(\nabla_{ad} - \nabla \right)$$

↳ Brunt-Väisälä frequency

for $N^2 < 0$ → growth → convective instability
$N > 0$ → oscillating solution → g-modes

$$\frac{d^2 \delta r}{dt^2} = -\frac{g}{H_p}(\nabla_{ad} - \nabla)\,\delta r \quad \Big|\cdot 2\frac{d\delta r}{dt}$$

$$\frac{d}{dt}\left(\frac{d\delta r}{dt}\right)^2 = +\frac{g}{H_p}(\nabla - \nabla_{ad})\frac{d\delta r^2}{dt} \quad \Big| \int dt$$

$$\Rightarrow \left(\frac{d\delta r}{dt}\right)^2 = \frac{g}{H_p}(\nabla - \nabla_{ad})\,\delta r^2$$

introduce mixing-length → element travels without being destroyed to the distance l, then it merges with surrounding. We set l so that $\overline{\delta r} = l/2$

→ mean convective velocity $\overline{\frac{d\delta r}{dt}} = \bar{v}$

then $\bar{v}^2 = \frac{g}{4H_p}(\nabla - \nabla_{ad})\,l^2$

energy flux: $F_c = \rho v^2 \cdot v \sim \rho\left[\frac{g}{4H_p}(\nabla - \nabla_{ad})\,l^2\right]^{3/2}$

deep in convection zone: $\frac{\nabla - \nabla_{ad}}{\nabla} \ll 1$
$\nabla > \nabla_{ad}$, but not much
⇒ slow velocities, time goes slow, all in local equilibrium
⇒ zone of efficient convection

under the surface $\frac{d\rho}{dr}$ steep decrease
⇒ large velocities, $\nabla \gg \nabla_{ad}$, speeds approach the speed of sound
⇒ inefficient convection
⇒ superadiabatic zone
(there is more than just the adiabatic convection which transports the energy)

in the mixing-length theory

$\boxed{l = \alpha H_p}$, $\alpha \sim 1$
↳ free parameter of the model
„universal constant"

Convection with radiative losses

the bubble irradiates during its rise

then $\underbrace{\nabla'}_{\text{real } \nabla} \neq \nabla_{ad}$; $\bar{v}^2 = \frac{g}{4H_p}(\nabla - \nabla')\,\ell^2$

temperature change within the bubble:

$$\Delta T = \Big[\underbrace{\Big(\frac{dT}{dr}\Big)'}_{\text{gradient inside the bubble}} - \underbrace{\frac{dT}{dr}}_{\text{surroundings}}\Big]\delta r$$

with respect to the surroundings

gradient inside the bubble (not adiabatic, because it irradiates)

we keep using $\alpha = \ell / H_p$

convective flux: $F_c = \underbrace{\Delta T \rho c_p}_{\text{energy content = heat}} \underbrace{\bar{v}}_{\text{speed}}$

energy content = heat ← calorimetric equation

$$(**) \quad \Delta T = \Big[\Big(\frac{dT}{dr}\Big)' - \frac{dT}{dr}\Big]\delta r = (\nabla - \nabla')\frac{T\,\delta r}{H_p}$$

$$\nabla' = -H_p \frac{d\ln T'}{dr}$$

$$\delta r = \ell/2, \quad \ell = \alpha H_p \Rightarrow \delta r = \frac{\alpha H_p}{2}$$

$$\Rightarrow \Delta T = (\nabla - \nabla')\frac{T\alpha}{2}$$

$$\Rightarrow F_c = \alpha \rho c_p T \bar{v} \frac{(\nabla - \nabla')}{2}$$

radiative loss from the bubble → from structural equations
→ equation of heat ballance

$$(*) \quad F_R' = -\frac{16\sigma T^3}{3\kappa\rho}\frac{\Delta T}{d} = \frac{8\alpha\sigma T^4}{3\kappa\rho d}(\nabla' - \nabla)$$

d ... distance at which $\Delta T \to 0$
→ it corresponds to the size of the cell

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total convective flux → adiabatic + correction

$$F_c = F_c^{ad} + F_c' = \alpha\rho c_p v T(\nabla - \nabla_{ad})/2 + \alpha\rho c_p v T(\nabla_{ad} - \nabla')/2$$

only splitting in two contributions

super-adiabacity obvious now

„convective deficite"

within the cell – equilibrium

$F_R' = F_C'$ [convective deficite is radiated out]

hence: $\frac{8\sigma T^4}{3\varkappa\rho d}(\nabla' - \nabla) = \rho c_p \bar{v} T (\nabla_{ad} - \nabla')/2$

$\hookrightarrow \bar{v} = \left[\frac{g}{4H_p}(\nabla - \nabla')\ell^2\right]^{1/2}$

total energy ballance

$F_R + F_C = \frac{L_\odot}{4\pi r^2}$

$$\frac{16\sigma T^4}{3\varkappa\rho H_p}\nabla + \alpha\rho c_p T \ell \sqrt{\frac{g}{4H_p}}\frac{(\nabla - \nabla')^{3/2}}{2} = \frac{L_\odot}{4\pi r^2}$$

$\frac{\Delta T}{d} = \frac{\nabla \, T d}{H_p} \cdot \frac{1}{d} = T\frac{\nabla}{H_p}$ analogy of (*) using (**)

two equations for ∇ and ∇', ∇_{ad} known

$\frac{dp}{p} = \gamma\frac{d\rho}{\rho}$; $\underbrace{p = A\rho T}_{\text{divide}} \Rightarrow dp = AT d\rho + A\rho\, dT$

$$\frac{dp}{p} = \frac{AT d\rho + A\rho\, dT}{A\rho T} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

$$\nabla_{ad} = \left(\frac{d\ln T}{d\ln p}\right)_{ad} = \frac{p}{T}\left(\frac{dT}{dp}\right)_{ad}$$

$$1 = \frac{p}{\rho}\frac{d\rho}{dp} + \frac{p}{T}\left(\frac{dT}{dp}\right)_{ad} \Rightarrow \frac{p}{T}\left(\frac{dT}{dp}\right)_{ad} = 1 - \underbrace{\frac{p}{\rho}\frac{d\rho}{dp}}_{1/\gamma} =$$

$$= 1 - 1/\gamma = \frac{\gamma - 1}{\gamma} = \nabla_{ad}$$

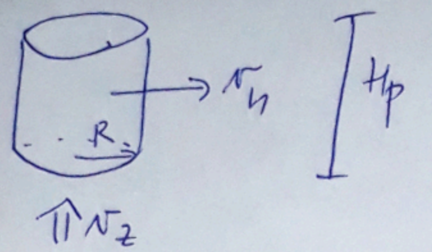
Pozn: Ledoux ~~...~~ parameter ~~...~~ of convective stability

$$A^* = \frac{1}{\gamma}\frac{d\ln p}{d\ln r} - \frac{d\ln\rho}{d\ln r}$$

$A^* < 0 \Leftarrow$ convective instability

The Sun: $\frac{dT}{dr}$ steep in outer layers $\rightarrow$ convection sets on to maintain the heat flux ($\sim T^3$)

Convective scales



continuity equation:

$$\pi R^2 v_z \rho \sim 2\pi R H_p \rho v_h$$

$$\Rightarrow R \sim 2 H_p \frac{v_h}{v_z}$$

estimates:

ballance of radiative losses and enthalpy flux

$H = U + pV$ ← enthalpy: energy content in the thermodynamical system

$$\sigma T_{eff}^4 \sim \rho v_z H$$

for hydrogen plasma: $H = 5/2\, kT + x\chi$

χ ... ionisation potential of hydrogen

x ... relative fraction of ionised hydrogen

$$\Rightarrow v_z = \frac{\sigma T_{eff}^4}{\rho H}$$

for $x \sim 0.1$ and solar values at the surface $v_z \sim 2$ km/s

horizontal speed – upper limit by sound speed in the photosphere

$c_s \sim 7$ km/s

$$\Rightarrow 2R = \underbrace{4\,\mathrm{Mm}}_{} \quad \text{for } H_p = 300\,\mathrm{km}$$

upper limit for the horizontal extend of convective scales on thermal-dissipation scale

→ depth dependence

$$R \sim 2H_p \frac{v_h}{v_z} = 2H_p \frac{c_s}{v_z} = 2H_p \frac{c_s}{\frac{\sigma T_{eff}^4}{\rho(5/2\, kT + x\chi)}} =$$

$$= 2H_p \frac{c_s\,(5/2\, kT + x\chi)\,\rho}{\sigma T_{eff}^4}$$

$$\frac{dR}{dr} = A\,\frac{dH_p}{dr} + B\,\frac{dc_s}{dr} + C\,\frac{dT}{dr} + D\,\frac{d\rho}{dr}$$

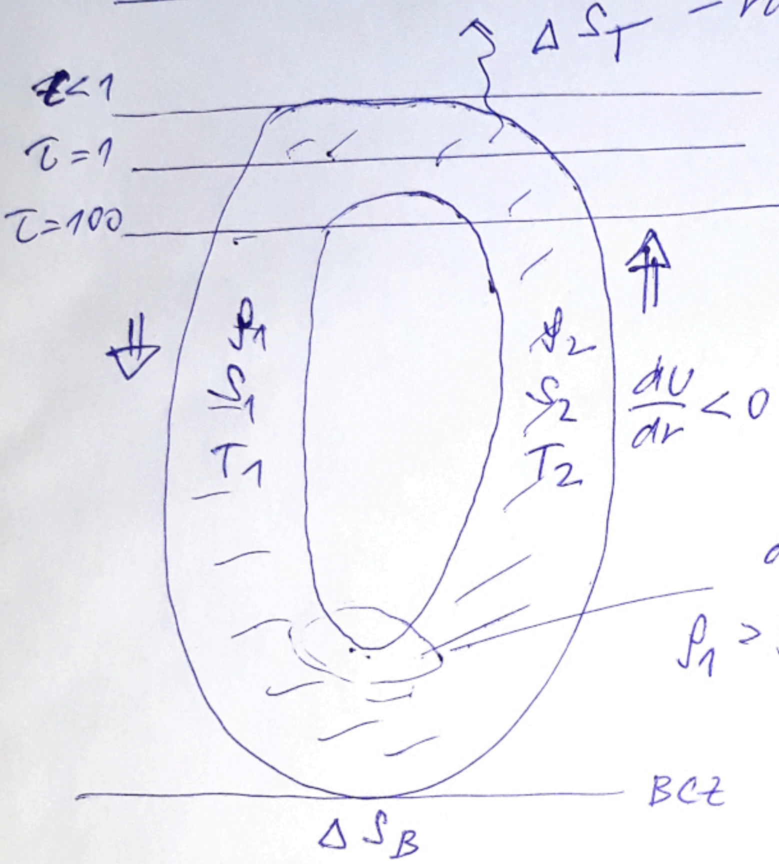
in the Sun: $\frac{\partial H_p}{\partial r} < 0$, $\frac{\partial c_s}{\partial r} < 0$, $\frac{\partial T}{\partial r} < 0$, $\frac{\partial \rho}{\partial r} < 0$

$$A, B, C, D > 0$$

$\Rightarrow \underline{\frac{\partial R}{\partial r} < 0}$ – with height the typical horizontal scale of convective cells decreases

$\boxed{\text{Convection}}$

ΔS_T – radiative losses



for $T_2 < T_{ion}$ → recombination, fast cooling of the material by photons

drag & mixing → S_2 rises

$\rho_1 > \rho_2$, $S_1 < S_2$, $T_1 < T_2$

$\Delta S_T \gg \Delta S_B$

motion directed from the surface, where entropy fluctuations are largest (larger than at BCZ)
most of the work is done by sinking plasma

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until 20 Mm depth → temperature rise from 4300 K → 143 000 K
density by 5.5 orders of mag.
pressure by 7 orders

20 Mm → BCZ – ↗ similarly fast ~~decrease~~ increase in temperature and pressure

$\Rightarrow$ all important at the surface!