Astronomy 112: The Physics of Stars

Class 7 Notes: Basics of Nuclear Fusion

In this class we continue the process of filling in the missing microphysical details that we need to make a stellar model. To recap, in the last two classes we computed the pressure of stellar material and the rate of energy transport through the star. These were two of the missing pieces we needed. The third, which we'll sketch out over the next two lectures, is the rate for nuclear reactions, and the energy that they generate.

I. Energetics

    A. Energy Release

All nuclear reactions fundamentally work by converting mass into energy. (In some ways the same could be said of chemical reactions, but for those the masses involved are so tiny as to not be worth worrying about.) The masses of the reactants involved therefore determine the energy released by the reaction.

Consider a reaction between two species that produced some other species

$$\mathcal{I}(\mathcal{A}_i, \mathcal{Z}_i) + \mathcal{J}(\mathcal{A}_j, \mathcal{Z}_j) \rightarrow \mathcal{K}(\mathcal{A}_k, \mathcal{Z}_k) + \mathcal{L}(\mathcal{A}_l, \mathcal{Z}_l),$$
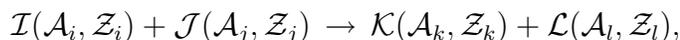
where as usual $\mathcal{Z}$ is the atomic number and $\mathcal{A}$ is the atomic mass number. At this point we must distinguish between atomic mass number and actual mass, so let $\mathcal{M}$ be the mass of a given species. The atomic mass number times m_{H} and the true mass are nearly identical, $\mathcal{M} \approx \mathcal{A}m_{\mathrm{H}}$, but not quite, and that small difference is the source of energy for the reaction. For the reaction we have written down, the energy released is

$$Q_{ijk} = (\mathcal{M}_i + \mathcal{M}_j - \mathcal{M}_k - \mathcal{M}_l)c^2,$$

i.e. the initial mass minus the final mass, multiplied by c^2.

To remind you, the rate per unit volume at which the reaction we have written down occurs is

$$\frac{\rho^2}{m_{\mathrm{H}}^2} \left(\frac{1}{1 + \delta_{ij}} \right) \frac{X_i X_j}{\mathcal{A}_i \mathcal{A}_j} R_{ijk},$$

where R_{ijk} is the rate coefficient. If each such reaction released an energy Q_{ijk}, then the rate of nuclear energy release per unit volume is simply given by this rate, multiplied by Q_{ijk}, and summed over all possible reactions:

$$\frac{\rho^2}{m_{\mathrm{H}}^2} \sum_{ijk} \left(\frac{1}{1 + \delta_{ij}} \right) \frac{X_i X_j}{\mathcal{A}_i \mathcal{A}_j} R_{ijk} Q_{ijk}.$$

The rate of nuclear energy release per unit mass is just this divided by the mass per volume ρ:

$$q_{\text{nuc}} = \frac{\rho}{m_{\text{H}}^2} \sum_{ijk} \left(\frac{1}{1 + \delta_{ij}} \right) \frac{X_i X_j}{\mathcal{A}_i \mathcal{A}_j} R_{ijk} Q_{ijk}.$$

If the reaction produces neutrinos, they will carry away some of the energy and escape the star, and thus the amount by which the star is heated will be reduced. However this loss is small in most stars under most circumstances.

B. Binding Energy per Nucleon

A very useful way to think about the amount of energy available in nuclear reactions is to compute the binding energy per nucleon. Suppose that we start with hydrogen, which consists of one proton of mass m_{H} (ignoring electrons), and we define that to have zero binding energy. Since binding energy is potential energy, we can do this, since we can choose the zero of potential energy to be anywhere.

Now consider some other element, with atomic mass number $\mathcal{A}$ and actual mass $\mathcal{M}$ per atom; and consider how much energy is released in the process of making that element from hydrogen. The exact reaction processes used don't matter, just the initial and final masses. Since atomic number is conserved, we must use $\mathcal{A}$ hydrogen atoms to make the new nucleus, so the difference between the final and initial mass is $\mathcal{M} - \mathcal{A}m_{\text{H}}$. We define the mass excess as this quantity multiplied by c^2:

$$\Delta M = (\mathcal{M} - \mathcal{A}m_{\text{H}})c^2.$$

This is just the difference in energy between the bound nucleus and the equal number of free protons. The name is somewhat confusing, since this is really an energy not a mass. The reason for the name is that in relativity one doesn't really need to distinguish between mass and energy. They're the same thing, just measured in different units.

A more useful quantity than this is the binding energy per nucleon, i.e. minus the mass excess divided by the number of nucleons (protons or neutrons) in the nucleus. The minus here is added so that the binding energy is positive if the nucleus is more strongly bound than the corresponding number of free nucleons. Thus we define the binding energy per nucleon as

$$-\frac{\Delta M}{\mathcal{A}} = \left(1 - \frac{\mathcal{M}}{\mathcal{A}m_{\text{H}}} \right) m_{\text{H}} c^2.$$

Since $\mathcal{M}$ and $\mathcal{A}$ can be determined experimentally, this quantity is fairly straightforward to measure. The results are very illuminating.

[Silde 1 – binding energy per nucleon]

This plot contains an enormous amount of information, and looking at it immediately explains a number of facts about stars and nuclear physics. To interpret this plot, recall that number of nucleons is conserved by nuclear reactions. Thus any nuclear reaction just involves taking a fixed number of nucleons and moving them

to the left or right on this plot. The energy released or absorbed in the process is just the number of nucleons involved multiplied by the change in binding energy per nucleon.

The first thing to notice about this plot is that there is a maximum at ^{56}Fe – iron-56. This is the most bound nucleus. At smaller atomic masses the binding energy per nucleon generally increases with atomic number, while at larger atomic masses it decreases. This marks the divide between fusion and fission reactions. At atomic masses below 56, energy is released by increasing the atomic number, so fusion is exothermic and fission is endothermic. At atomic number above 56, energy is released by decreasing the atomic number, so fission is exothermic and fission is endothermic.

Second, notice that the rise is very sharp at small atomic number. This means that fusing hydrogen into heavier things is generally the most exothermic reaction available, and that it releases far more energy per nucleon than later stages of fusion, say helium into carbon. This has important implications for the fate of stars that exhaust their supply of hydrogen.

Third, notice that there are several local maxima and minima at small atomic number. ^{4}He is a maximum, as are ^{12}C and ^{16}O. There is a good reason that helium, carbon, and oxygen are the most common elements in the universe after hydrogen: they are local maxima of the binding energy, which means that they are the most strongly bound, stable elements in their neighborhood of atomic number. Conversely, lithium is a minimum. For this reason nuclear reactions in stars destroy lithium, and they do not produce it. Essentially all the lithium there is in the universe was made in the big bang, and stars have been destroying it ever since.

Finally, notice that these are big numbers as far as the energy yield. The scale on this plot is MeV per nucleon. In terms of more familiar units, 1 MeV per H mass corresponds to 10^{18} erg g^{-1}, or roughly 22 tons of TNT per gram of hydrogen fuel.

II. Reaction Rates

A. The Coulomb Barrier

The binding energy curve tells us the amount of energy available from nuclear reactions, but not the rates at which they occur. Given that the reaction for fusing hydrogen to helium is highly exothermic, why doesn't the reaction happen spontaneously at room temperature? The answer is the same as the reason that coal doesn't spontaneously combust at room temperature: the reaction has an activation energy, and that energy is quite high.

To understand why, consider the potential energy associated with two nuclei of charge $\mathcal{Z}_i$ and $\mathcal{Z}_j$ separated by a distance r. The Coulomb (electric) potential

energy is

$$U_{\rm C} = \frac{\mathcal{Z}_i \mathcal{Z}_j e^2}{r} = \mathcal{Z}_i \mathcal{Z}_j \frac{1.4 \text{ MeV}}{r/\text{fm}},$$

where 1 fm $= 10^{-13}$ cm $= 10^{-15}$ m. Since this is positive, the force between the protons is repulsive, as it should be.

In addition to that positive energy, there is a negative energy associated with nuclear forces. The full form of the proton-proton force is complicated, but we can get an idea of its behavior by noting that, at larger ranges, it is mediated by the exchange of virtual mesons such as pions. Because these particles have mass, their range is limited by the Heisenberg uncertainty principle: they can only exist for a short time, and they only exert significant force at distances they can reach within that time. Specifically, the uncertainty principle tells us that

$$\Delta E \, \Delta t \geq \frac{\hbar}{2}$$

If the particle travels at the maximum possible speed of c, its range is roughly

$$r \sim c\Delta t \sim \frac{c\hbar}{E},$$

where E is the rest energy of the particle being exchanged. For pions, which mediate the proton-proton force, $\Delta E = 135$ MeV or 140 MeV, depending on whether they are neutral or charged. Plugging this in for ΔE gives $r \sim 1$ fm. Thus the nuclear force is negligible at distances greater than ~ 1 fm. Within that range, however, the nuclear force is dominant. Potentials arising from exchanges of massive particles like this are called Yukawa potentials, and they have the form

$$U_{\rm Y} = -g^2 \frac{e^{-r/\lambda}}{r},$$

where g is a constant and $\lambda = c\hbar/E$ is the range of the force. This is only an approximation to the true potential energy, but it is reasonably good one at large ranges.

The total potential is the sum of the Yukawa and Coulomb potentials. The functional form of this potential is something like a $1/r$ rise that is cut off by a sharp decrease at small radii. This slide shows an example for an important reaction: ^{12}C $+ \alpha$, which has $\mathcal{Z}_i = 6$ and $\mathcal{Z}_j = 2$.

[Slide 2 – Coulomb barrier for ^{12}C $+ \alpha$ reaction]

For the reaction to proceed, the two particles must get close enough to one another to reach the region where the potential drops, and the force becomes attractive. If they do not, they will simply bounce off one another without reacting. This is called the Coulomb barrier, and it applies to chemical as well as nuclear reactions. The existence of the Coulomb barrier means that there is a minimum relative velocity the particles must have in order for the reaction to go, which we can

calculate from the height of the Coulomb barrier. This is much like rolling a ball up a steep hill with a peak – there is a minimum velocity with which you must roll the ball if you want it to reach the top of the hill.

Suppose that the potential follows the Coulomb form until some minimum radius $r_0 \sim 1$ fm, then suddenly drops at smaller radii. The maximum potential energy is

$$U_C = \frac{\mathcal{Z}_i \mathcal{Z}_j e^2}{r_0} = \mathcal{Z}_i \mathcal{Z}_j \frac{1.4 \text{ MeV}}{r_0/\text{fm}}.$$

The minimum relative velocity of the particles is given by the condition that the kinetic energy in the center of mass frame exceed this value:

$$\frac{1}{2} \mu_{\text{red}} m_{\text{H}} v^2 \geq U_c,$$

where $\mu_{\text{red}} m_{\text{H}}$ is the reduced mass.

A more useful calculation than this is to ask what temperature the gas must have such that the typical collision is at sufficient velocity for the reaction to occur. The typical collision energy is

$$\frac{1}{2} \mu_{\text{red}} m_{\text{H}} v^2 = \frac{3}{2} k_B T,$$

so setting this equal to U_C and solving gives

$$T \geq \frac{2 \mathcal{Z}_i \mathcal{Z}_j e^2}{3 k_B r_0} = 1.1 \times 10^{10} \text{ K } \frac{\mathcal{Z}_i \mathcal{Z}_j}{r_0/\text{fm}}.$$

Thus the typical particle does not have enough energy to penetrate the Coulomb barrier until the temperature is $\sim 10^{10}$ K for proton-proton reactions, and even higher temperatures for higher atomic numbers. This is much higher than the temperatures for stars' centers than we estimated earlier in the class. You might think that it's not a problem because some particles move faster than the average, and thus are going fast enough to penetrate the Coulomb barrier. You will show on your homework that this solution doesn't work. At the temperature of $\sim 10^7$ K in the center of the Sun, if this calculation is correct then fusion should not be possible.

B. Quantum Tunneling

The resolution to this problem lies in the phenomenon of quantum tunneling. The calculation we just did is based on classical physics, and predicts that no nuclei will get within r_0 of one another unless they reach a high enough velocity to overcome the Coulomb barrier. However, in quantum mechanics there is a non-zero probability of finding the particle inside r_0 even if it does not have enough energy to break the Coulomb barrier. This phenomenon is known as tunneling, because it is like the particle takes a tunnel through the peak rather than going over it.

We can make a crude estimate of when tunneling will occur using wave-particle duality. Recall that each proton can be thought of as a wave whose wavelength is dictated by the uncertainty principle. The wavelength associated with a particle of momentum p is

$$\lambda = \frac{h}{p}.$$

This is known as the particle's de Broglie wavelength.

As a rough estimate of when quantum tunneling might allow barrier penetration, we can estimate that the two particles must be able to get within one de Broglie wavelength of one another. This in turn requires that the kinetic energy of the particles be equal to their Coulomb potential energy at a separation of one de Broglie wavelength:

$$\frac{\mathcal{Z}_i \mathcal{Z}_j e^2}{\lambda} = \frac{1}{2}\mu_{\text{red}} m_{\text{H}} v^2 = \frac{p^2}{2\mu_{\text{red}} m_{\text{H}}} = \frac{h^2}{2\mu_{\text{red}} m_{\text{H}} \lambda^2}$$

Solving this for λ, we find that barrier penetration should occur is the particles are able to get within a distance

$$\lambda = \frac{h^2}{2\mu_{\text{red}} m_{\text{H}} \mathcal{Z}_i \mathcal{Z}_j e^2}.$$

of one another.

To figure out the corresponding temperature, we can just evaluate our result from the classical problem using λ in place of r_0:

$$T \geq \frac{2\mathcal{Z}_i \mathcal{Z}_j e^2}{3k_B \lambda} = \frac{4\mathcal{Z}_i^2 \mathcal{Z}_j^2 e^4 \mu_{\text{red}} m_{\text{H}}}{3h^2 k_B} = 9.6 \times 10^6 \text{ K } \mathcal{Z}_i^2 \mathcal{Z}_j^2 \left(\frac{\mu_{\text{red}}}{1/2}\right).$$

Thus proton-proton reactions, which have $\mathcal{Z}_i = \mathcal{Z}_j = 1$ and $\mu_{\text{red}} = 1/2$, should begin to occur via quantum tunneling at a temperature of $\sim 10^7$ K, much closer to the temperatures we infer in the center of the Sun.

C. The Gamow Peak

Having seen that quantum effects are important, we will now try to perform a more rigorous calculation of the reaction rate. Consider reactions between two nuclei with number densities n_i and n_j in a gas at temperature T. In order to compute the reaction rate, we need to know the rate at which these nuclei collide with enough energy to tunnel through the Coulomb barrier. That's what we'll calculate now.

The first step is to compute the rate at which particles strike one another closely enough to interact. This is very much like calculating the pressure. We consider a particle, and we want to know how often other particles run into it. If we had a beam of particles of density n and velocity v, and the target particle had a cross-sectional area σ, the impact rate would be $n\sigma v$. Note that this formula is almost exactly like the one describing the rate at which particles strike the wall of a vessel, which we used to compute pressures.

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particle density n

particle velocity v

target area σ
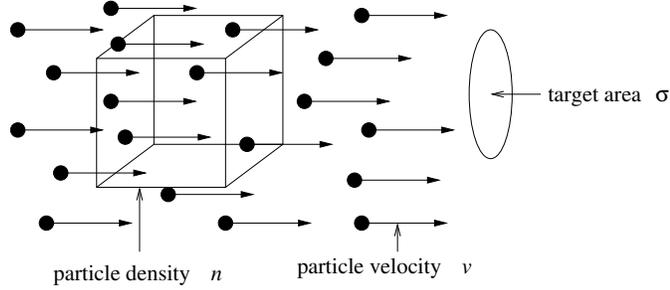
In reality the particle in question isn't a solid target with a fixed area. We're interested in interactions that lead to reactions, which require that the collision be close enough to allow the nuclei to tunnel through the Coulumb barrier, but also require that the interaction have enough energy to make such tunneling possible. A direct bullseye at a very low energy won't lead to a reaction, so the cross-section at very low energies is basically zero. However, we can still extend the analogy of shooting a beam of particles at a target by defining the cross-section at energy E. Let $dN_{\text{reac}}(E)/dt$ be the number of reactions per time interval dt produced by shooting a beam of particles of density n at velocity v at a target nucleus. We define the cross-section $\sigma(E)$ via the relation

$$\frac{dN_{\text{reac}}(E)}{dt} = n\sigma(E)v(E).$$

Next we want to generalize from a the case of a beam to the case of a thermal gas where not all particles have the same energy. We proved a few classes ago that the momentum distribution of particles of mass m at temperature T is

$$\frac{dn}{dp} = \frac{4n}{\pi^{1/2}(2mk_BT)^{3/2}}p^2 e^{-p^2/(2mk_BT)}.$$

Since we're interested in particle energies, we'll change this to a distribution over energy instead of momentum. Since $E = p^2/(2m)$, or $p = \sqrt{2mE}$, we have

$$\frac{dn}{dE} = \frac{dn}{dp}\frac{dp}{dE} = \frac{4n}{\pi^{1/2}(2mk_BT)^{3/2}}p^2 e^{-p^2/(2mk_BT)} \cdot \sqrt{\frac{m}{2E}} = \frac{2n}{\pi^{1/2}(k_BT)^{3/2}}E^{1/2}e^{-E/k_BT}.$$

Note that this only applies to non-relativistic particles, since we used $E = p^2/(2m)$ instead of $E = pc$. However, nuclei are generally always non-relativistic, except in neutron stars.

In this case, the number of reactions dN per time interval dt that a given target nucleus undergoes is given by integrating over the possible energies of the impacting particles. In particular, the number of reactions per unit time for a particle of species i due collisions with particles of species j is

$$\frac{dN_i}{dt} = \int_0^\infty \sigma(E)v(E)\frac{dn_j}{dE}\,dE.$$

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Since the velocity that matters here is the relative velocity, we have to compute it in terms of the reduced mass: $v(E) = \sqrt{2E/\mu_{\text{red}}m_{\text{H}}}$, where $\mu_{\text{red}}m_{\text{H}} = m_i m_j/(m_i + m_j)$. Finally, if we want to know the number of reactions per unit time in a given volume of gas, we just have to multiply this by the number of target nuclei per unit volume, n_i, and divide by $(1 + \delta_{ij})$ to avoid double-counting. This gives

$$\frac{dn_{\text{reac}}}{dt} = \frac{n_i}{(1 + \delta_{ij})} \int_0^\infty \sigma(E) v(E) \frac{dn_j}{dE} \, dE.$$

Recall that we defined the rate coefficient R_{ijk} so that the reaction rate is $R_{ijk}n_i n_j$ for different species, or $R_{ijk}n_i^2/2$ for two of the same species. Thus the rate coefficient is

$$
\begin{aligned}
R_{ijk} &= \frac{(1 + \delta_{ij})}{n_i n_j} \frac{dn_{\text{reac}}}{dt} \\
&= \frac{2}{\pi^{1/2}} \frac{1}{(k_B T)^{3/2}} \int_0^\infty \sigma(E) v(E) E^{1/2} e^{-E/k_B T} \, dE \\
&= \frac{1}{(\pi \mu_{\text{red}} m_{\text{H}})^{1/2}} \left(\frac{2}{k_B T} \right)^{3/2} \int_0^\infty \sigma(E) E e^{-E/k_B T} \, dE
\end{aligned}
$$

The final remaining step is to figure out the cross-section $\sigma(E)$ at energy E. Computing this in general is quite difficult, and often laboratory measurements are required to be sure of exact values. However, we can get a rough idea of how $\sigma(E)$ varies with energy based on general quantum-mechanical principles. The first such principle is that particles should interact when they come within distances that are comparable to their de Broglie wavelengths – a higher energy particles has a smaller wavelength, and thus represents a smaller target. Thus we expect that

$$\sigma(E) \propto \lambda^2 = \frac{h^2}{p^2} \propto \frac{1}{E}.$$

The second principle is that nuclear reactions like the ones we are interested in require tunneling through the Coulomb barrier. A quantum mechanical calculation of the probability that tunneling will occur shows that it is proportional to

$$e^{-2\pi^2 U_C/E},$$

where U_C is the height of the Coulomb barrier at a distance of one de Broglie wavelength. You will see this calculation in your quantum mechanics class, and I will not go through it here. In terms of the energy, the Coulomb barrier U_C is

$$U_C = \frac{\mathcal{Z}_i \mathcal{Z}_j e^2}{\lambda} = \frac{\mathcal{Z}_i \mathcal{Z}_j e^2 p}{h} = \frac{\mathcal{Z}_i \mathcal{Z}_j e^2}{h} \sqrt{2\mu_{\text{red}} m_{\text{H}} E},$$

so the exponential factor is

$$2\pi^2 \frac{U_C}{E} = 2^{3/2} \pi^2 \frac{\mu_{\text{red}}^{1/2} m_{\text{H}}^{1/2} \mathcal{Z}_i \mathcal{Z}_j e^2}{h} E^{-1/2} \equiv b E^{-1/2},$$

where
$$b = 2^{3/2}\pi^2 \frac{\mu_{\text{red}}^{1/2} m_{\text{H}}^{1/2} \mathcal{Z}_i \mathcal{Z}_j e^2}{h} = 0.0013 \mu_{\text{red}}^{1/2} \mathcal{Z}_i \mathcal{Z}_j \ (\text{erg})^{1/2}.$$

Thus we also expect to have $\sigma \propto e^{-bE^{-1/2}}$. Note that the factor b depends only on the charges and masses of the nuclei involved in the reaction. It is therefore constant for any given reaction.

Combining the two factors our analysis reveals, we define
$$\sigma(E) \equiv \frac{S(E)}{E} e^{-bE^{-1/2}},$$

where $S(E)$ is, ideally, either a constant or a function that varies only very, very weakly with E. Plugging all this in, the reaction rate coefficient is
$$R_{ijk} = \frac{1}{(\pi \mu_{\text{red}} m_{\text{H}})^{1/2}} \left(\frac{2}{k_B T}\right)^{3/2} \int_0^\infty S(E) e^{-bE^{-1/2}} e^{-E/k_B T} \, dE.$$

It is instructive to look at the behavior of the two exponential factors, $e^{-bE^{-1/2}}$ and $e^{-E/k_B T}$. Clearly the first function increases as E increases, while the second one decreases as E increases. We therefore expect their product to reach a maximum at some intermediate energy. In fact, we can compute the maximum analytically, by taking the derivative and setting in equal to zero:
$$
\begin{aligned}
0 &= \frac{d}{dE}\left(e^{-bE^{-1/2}} e^{-E/k_B T}\right) \\
&= \frac{d}{dE} e^{-(E/k_B T + bE^{-1/2})} \\
&= -\left(\frac{E}{k_B T} + bE^{-1/2}\right)\left(\frac{1}{k_B T} - \frac{b}{2E^{3/2}}\right) e^{-(E/k_B T + bE^{-1/2})} \\
E_0 &= \left(\frac{b k_B T}{2}\right)^{2/3} \\
&= 1.22 \left[\mathcal{Z}_i^2 \mathcal{Z}_j^2 \mu_{\text{red}} \left(\frac{T}{10^6 \text{ K}}\right)^2\right]^{1/3} \text{ keV},
\end{aligned}
$$

where E_0 is the energy at the maximum. This maximum is known as the Gamow peak, after George Gamow, who discovered it in 1928. The plot shows the Gamow peak for proton-proton interactions at $T = 1.57 \times 10^7$ K, the Sun's central temperature.

[Slide 3 – the Gamow peak]

If we let $x = E/E_0$, then we can rewrite the reaction rate coefficient as
$$
\begin{aligned}
R_{ijk} &= \frac{E_0}{(\pi \mu_{\text{red}} m_{\text{H}})^{1/2}} \left(\frac{2}{k_B T}\right)^{3/2} \int_0^\infty S(x) \exp\left[-\left(\frac{b^2}{4 k_B T}\right)^{1/3}\left(x + \frac{2}{x^{1/2}}\right)\right] dx \\
&= \left[2^{11}\pi^5 \frac{\mathcal{Z}_i^4 \mathcal{Z}_j^4 e^8}{\mu_{\text{red}} m_{\text{H}} h^4 (k_B T)^5}\right]^{1/6} \int_0^\infty S(x) \exp\left[-\left(\frac{b^2}{4 k_B T}\right)^{1/3}\left(x + \frac{2}{x^{1/2}}\right)\right] dx
\end{aligned}
$$

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To get a sense of how narrowly peaked this function is, it is helpful to evaluate the factor $[b^2/(4k_BT)]^{1/3}$ for some typical examples. If we consider proton-proton interactions (so $\mathcal{Z}_i = \mathcal{Z}_j = 1$ and $\mu_{\mathrm{red}} = 1/2$) at the Sun's central temperature of 1.57×10^7 K, then we have

$$b = 8.8 \times 10^{-4} \ (\mathrm{erg})^{1/2} \ \text{and} \ \left(\frac{b^2}{4k_BT}\right)^{1/3} = 4.5.$$

Evaluating the function $e^{-4.5(x+2/x^{1/2})}$ shows that for $x = 3$ (i.e. at energies three times the peak), it is a factor of 180 lower than it is at peak. For $x = 1/3$ (i.e. at energies three times below the peak), it is 35 times smaller than it is at peak. Thus the reaction rate is strongly dominated by energies near the peak, with energies different from the peak by even as little as a factor of 3 contributing negligibly.

When we are near the peak, i.e. $x \approx 1$, the reaction rate varies exponential with the quantity $[b^2/(k_BT)]^{1/3}$. This means that the reaction rate is extremely sensitive to temperature. For this reason, we often think of nuclear reactions as having a threshold temperature at which they turn on. This threshold temperature clearly increases with nuclear charge: since $b \propto \mathcal{Z}_i\mathcal{Z}_j$, and the reaction rate depends on b^2/T, we expect the temperature needed to ignite a particular reaction to vary as $\mathcal{Z}_i^2\mathcal{Z}_j^2$. Thus higher $\mathcal{Z}$ nuclei require progressively higher temperatures to fuse.

Of course we still have not assigned a value of $S(E)$ near the Gamow peak. We have only said that we expect it to be nearly constant. Its actual value depends on the reaction in question and the type of physics it involves, and must be obtained either by laboratory measurement, theoretical quantum calculation, or a combination of both. Unfortunately these values sometimes have significant uncertainties. In a star, reactions can occur at an appreciable rate at relatively low temperatures because the density is high – recall that the reaction rate per unit volume varies as $n_i n_j$. In a laboratory, we have to work with much lower densities, and as a result the reaction rates at the temperatures found in stars are often unobservably small. Instead, we are forced to make measurements at higher temperatures and extrapolate.

D. Temperature Dependence of Reaction Rates

It is often helpful to know roughly how the reaction rate varies with temperature when one is near the ignition temperature. To find that out, we can approximately evaluate the integral in the formula for the rate coefficient. As a first step in this approximation, we neglect any variation in the $S(E)$ factor across the Gamow peak, and simply set it equal to a constant value $S(E_0)$. Thus the reaction rate coefficient is approximately

$$R_{ijk} = \frac{1}{(\pi\mu_{\mathrm{red}}m_{\mathrm{H}})^{1/2}} \left(\frac{2}{k_BT}\right)^{3/2} S(E_0) \int_0^\infty \exp\left(-\frac{E}{k_BT} - \frac{b}{E^{1/2}}\right) dE.$$

The maximum value of the integrand occurs when $E = E_0$, and is given by

$$I_{\max} \equiv \exp\left(-\frac{3E_0}{k_B T}\right) \equiv e^{-\tau},$$

where we define

$$\tau = \frac{3E_0}{k_B T} = 42.46 \left[\mathcal{Z}_i^2 \mathcal{Z}_j^2 \mu_{\text{red}} \left(\frac{T}{10^6 \text{ K}}\right)^{-1}\right]^{1/3}$$

The second step in the approximation is to approximate the exponential factor in the integral by a Gaussian of width Δ:

$$\exp\left(-\frac{E}{k_B T} - \frac{b}{E^{1/2}}\right) \approx I_{\max} \exp\left[-\left(\frac{E - E_0}{\Delta/2}\right)^2\right].$$

The width Δ is generally chosen by picking the value such that the second derivatives of the exact and approximate forms for the integrand are equal at $E = E_0$. A little algebra shows that this gives

$$\Delta = \frac{4}{\sqrt{3}} (E_0 k_B T)^{1/2}.$$

The approximation is reasonably good. The graph shown is for two protons at a temperature of 1.6×10^7 K.

[Slide 4 – Gaussian approximation to the Gamow peak]

The final step in the approximation is to change the limits of integration from 0 to ∞ to $-\infty$ to ∞. This is not a bad approximation because the vast majority of the power in the Gaussian occurs at positive energies, and if the limits are $-\infty$ to ∞, the integral can be done exactly:

$$\int_{-\infty}^{\infty} \exp\left[-\left(\frac{E - E_0}{\Delta/2}\right)^2\right] dE = \frac{\sqrt{\pi}}{2} \Delta.$$

With this approximation complete, we can write the reaction rate coefficient as

$$\begin{aligned} R_{ijk} &= \frac{1}{(\pi \mu_{\text{red}} m_H)^{1/2}} \left(\frac{2}{k_B T}\right)^{3/2} S(E_0) I_{\max} \frac{\sqrt{\pi}}{2} \Delta \\ &= I_{\max} \left(\frac{2}{\mu_{\text{red}} m_H}\right)^{1/2} \frac{\Delta}{(k_B T)^{3/2}} S(E_0). \end{aligned}$$

We can rewrite this in terms of τ by substituting in for Δ and $k_B T$ in terms of τ. Doing so and simplifying a great deal produces

$$R_{ijk} = \frac{4}{3^{5/2} \pi^2} \frac{h}{\mu_{\text{red}} m_H \mathcal{Z}_i \mathcal{Z}_j e^2} S(E_0) \tau^2 e^{-\tau}.$$

All the temperature-dependence is encapsulated in the $\tau^2 e^{-\tau}$ term. The factor τ itself varies as

$$\tau \propto \frac{E_0}{T} \propto T^{-1/3}.$$

It is often useful to approximate the reaction rate as a powerlaw in T, i.e. to set $R_{ijk} \propto T^\nu$ for some power ν. Obviously the relationship is not a powerlaw in general, since there is an exponential in τ. However, we can approximate the behavior as a powerlaw if we are in the vicinity of a particular temperature T_0, near which $\tau = \tau_0 (T/T_0)^{-1/3}$. To understand what this entails, recall that a powerlaw is just a straight line in a log-log plot. In effect, fitting to a powerlaw is just the same as computing the slope at some point in the log-log plot. Thus we have

$$\nu = \frac{d \ln R_{ijk}}{d \ln T}$$

Since $R_{ijk} \propto \tau^2 e^{-\tau}$,

$$\ln R_{ijk} = 2 \ln \tau - \tau + \text{const} = -\frac{2}{3} \ln T - \tau_0 \left(\frac{T}{T_0} \right)^{-1/3} + \text{const}$$

Taking the derivative:

$$
\begin{aligned}
\nu = \frac{d \ln R_{ijk}}{d \ln T} &= -\frac{2}{3} - \tau_0 T_0^{1/3} \frac{d}{d \ln T} T^{-1/3} \\
&= -\frac{2}{3} - \tau_0 T_0^{1/3} T \frac{d}{dT} T^{-1/3} \\
&= -\frac{2}{3} + \frac{\tau_0 T_0^{1/3}}{3 T^{1/3}} \\
&= \frac{\tau}{3} - \frac{2}{3}
\end{aligned}
$$

This lets us approximate the behavior of R_{ijk} as a powerlaw:

$$R_{ijk} = R_{0,ijk} T^{(\tau-2)/3}.$$

We will use this in the next class to evaluated several of the important reactions inside stars. Given such a powerlaw fit, we can come up with an equivalent one for the rate of nuclear energy generation per unit mass when the gas temperature is near the ignition temperature for a given reaction:

$$q_{\text{nuc}} = \frac{\rho}{m_{\text{H}}^2} \sum_{ijk} \left(\frac{1}{1 + \delta_{ij}} \right) \frac{X_i X_j}{\mathcal{A}_i \mathcal{A}_j} R_{ijk} Q_{ijk} = \rho \sum_{ijk} \left(\frac{1}{1 + \delta_{ij}} \right) \frac{X_i X_j}{\mathcal{A}_i \mathcal{A}_j} q_{0,ijk} T^{p_{ijk}},$$

where $q_{0,ijk}$ and p_{ijk} are constants for a given reaction, i.e. they do not depend on gas density, element abundances, or gas temperature, as long as the temperature is near the ignition temperature.

E. Resonances and Screening

The simple model we have just worked out is reasonably good for many reactions of importance in stars, but it omits a number of complications, two of which we will discuss briefly.

First, the assumption that $S(E)$ varies weakly with E over the Gamow peak is not always valid. The most common way for the assumption to fail is if there is a resonance, which means that the energy of the collision corresponds closely to the energy of an excited state of the final product nucleus. If this happens, the cross section increases dramatically in a narrow range of energies, and $S(E)$ becomes sharply peaked. While none of the reactions involved in hydrogen burning in main sequence stars are resonant, some of the important reactions that occur in more evolved stars are. Resonances can enhance the reaction rate by orders of magnitude compared to what our our simple model would suggest.

A second complication is screening. Our calculation of the Coulomb barrier was based on the potential of two nuclei of charge $\mathcal{Z}_i$ and $\mathcal{Z}_j$ interacting with one another. However, this ignores the presence of electrons. For neutral atoms, the electric potential drops to zero for distances greater than a few angstroms, because the nucleus is surrounded by a cloud of electrons of equal and opposite charge. From a point outside the cloud, the net charge seen is zero, because the electronic and nuclear charges cancel – the electrons screen the nucleus. This is why neutral atoms do not violently repel one another.

In the fully ionized plasma inside a star electrons are not bound to atoms, and they float about freely. However, they are still attracted to the positively charged nuclei, and thus they tend to cluster around them, partly screening them. This effect reduces the Coulomb barrier. Screening is strongest at lower temperatures, since when $k_B T$ is smaller compared to the electric potential energy, electrons tend to to cluster more tightly around nuclei. This effect can enhance reaction rates for turning H into He by $\sim 10 - 50\%$ compared to the results of our naive calculation.