Helioseismology

resonance for p-modes

$$\int_{r_t}^{R_\odot} k_r \, dr = \pi (n+\alpha) = \int_{r_t}^{R_\odot} \left[\frac{\omega^2}{c^2} - \frac{l(l+1)}{r^2} \right]^{1/2} dr \quad \Big| \cdot \frac{1}{\omega}$$

$$\int_{r_t}^{R_\odot} \left(\frac{r^2}{c^2} - \frac{l(l+1)}{\omega^2} \right)^{1/2} \frac{dr}{r} = \frac{\pi (n+\alpha)}{\omega}$$

r_t ~ depends on $\frac{\sqrt{l(l+1)}}{\omega}$ $\qquad \frac{c(r_t)}{r_t} = \frac{\omega}{\sqrt{l(l+1)}}$

$\Rightarrow$ integrand on the left is a function of

$$\frac{\omega}{\sqrt{l(l+1)}}$$

$$F\left[\frac{l(l+1)}{\omega} \right] = \frac{\pi (n+\alpha)}{\omega}$$

$\rightarrow$ 2-D dispersion relation for $\omega = \omega(n, l)$ collapses to 1-D relation between $\frac{\sqrt{l(l+1)}}{\omega}$ and $\frac{n+\alpha}{\omega}$

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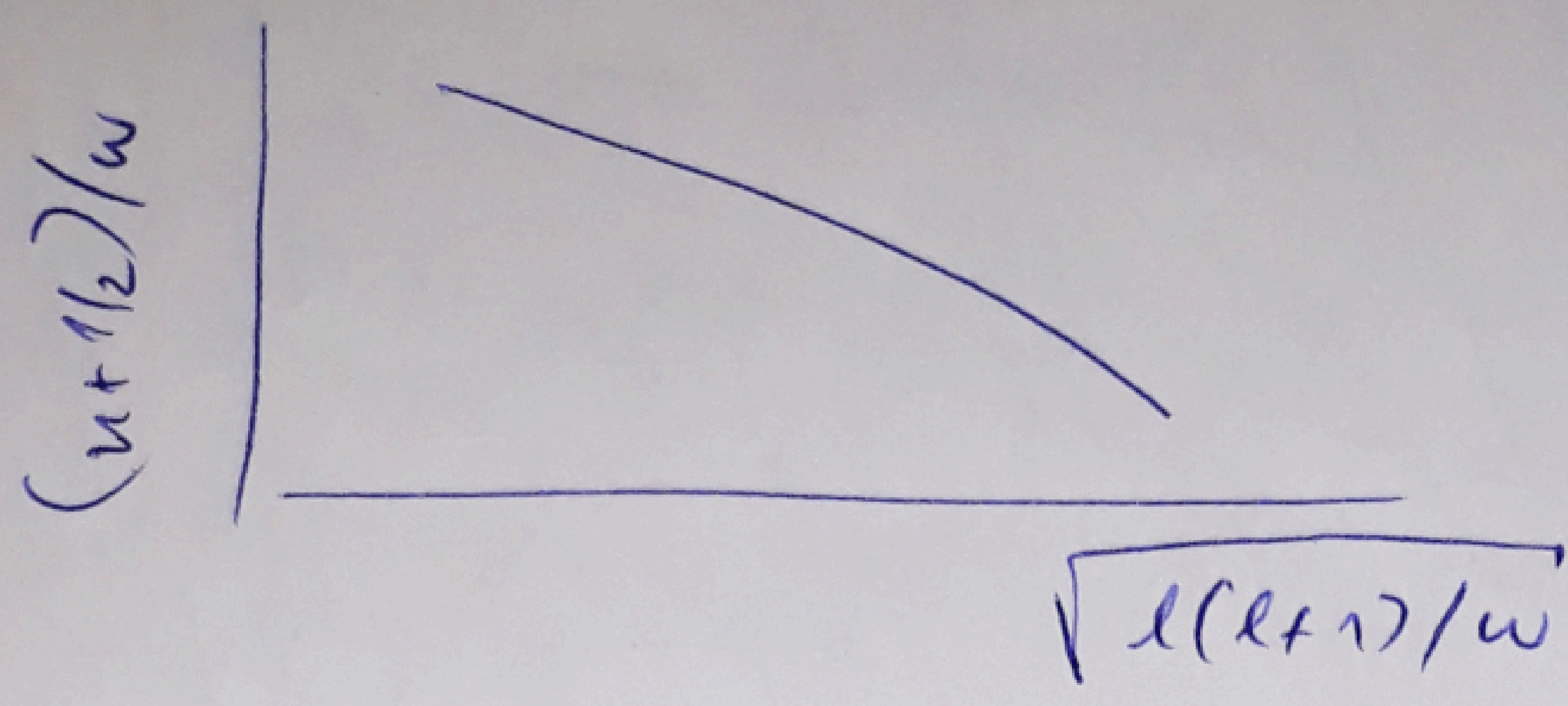
→ Duvall's law



$\boxed{\text{Helioseismic inversion}}$

- find a correction to solar model that minimizes the difference between measured and computed frequencies

$c \to c + \delta c$ perturbation in sound speed

$\omega \to \omega + \delta\omega$ causes perturbation in frequencies

$$\int_{r_t}^{R_\odot} \left[\frac{(\omega + \delta\omega)^2}{(c + \delta c)^2} - \frac{l(l+1)}{r^2} \right]^{1/2} dr = \pi (n + \alpha)$$

$$l(l+1) \equiv L^2$$

expansion about $\frac{\delta\omega}{\omega}$ and $\frac{\delta c}{c}$, it should equal

$$\delta\omega: \left[\left(\frac{\omega + \delta\omega}{c} \right)^2 - \frac{L^2}{r^2} \right]^{1/2} = \left[\frac{\omega^2}{c^2} \left(1 + \frac{\delta\omega}{\omega} \right)^2 - \frac{L^2}{r^2} \right]^{1/2} =$$

$$= \left| (1+x)^2 \sim 1 + 2x \cancel{+x^2} \right| = \left[\frac{\omega^2}{c^2} \left(1 + \frac{2\delta\omega}{\omega} \right) - \frac{L^2}{r^2} \right]^{1/2} =$$

$$= \left[\frac{\omega^2}{c^2} - \frac{L^2}{r^2} + \frac{2\delta\omega\,\omega}{c^2} \right]^{1/2} = \sqrt{\frac{\omega^2}{c^2} - \frac{L^2}{r^2}} \left[1 + \frac{2 \cdot \frac{\delta\omega\,\omega}{c^2}}{\frac{\omega^2}{c^2} - \frac{L^2}{r^2}} \right]^{1/2} =$$

$$= \left| \sqrt{1+x} \sim 1 + \frac{x}{2} \right| = \underbrace{\sqrt{\frac{\omega^2}{c^2} - \frac{L^2}{r^2}}}_{A} + \frac{\frac{\delta\omega\,\omega}{c^2}}{\sqrt{\frac{\omega^2}{c^2} - L^2/r^2}} =$$

$$= A + \frac{\frac{\delta\omega\,\omega}{c^2}}{\frac{\omega}{c} \left(1 - \frac{L^2 c^2}{r^2 \omega^2} \right)^{1/2}} = A + \frac{\delta\omega}{c} \frac{1}{\left(1 - \frac{L^2 c^2}{r^2 \omega^2} \right)^{1/2}}$$

$$\delta c: \left[\left(\frac{\omega}{\delta c + c}\right)^2 - \frac{L^2}{r^2}\right]^{1/2} = \left[\frac{\omega^2}{c^2}\left(\frac{1}{1+\frac{\delta c}{c}}\right)^2 - \frac{L^2}{r^2}\right]^{1/2} =$$

$$= \left[\frac{\omega^2}{c^2}\,\frac{1}{1+\frac{2\delta c}{c}} - \frac{L^2}{r^2}\right]^{1/2} = \left|\frac{1}{1+x} \sim 1-x\right| =$$

$$= \left[\frac{\omega^2}{c^2}\left(1-\frac{2\delta c}{c}\right) - \frac{L^2}{r^2}\right]^{1/2} = \left[\frac{\omega^2}{c^2} - \frac{L^2}{r^2} - \frac{2\delta c\,\omega^2}{c^3}\right]^{1/2} =$$

$$= \sqrt{\frac{\omega^2}{c^2} - \frac{L^2}{r^2}}\left(1 - \frac{\frac{2\delta c\,\omega^2}{c^3}}{\frac{\omega^2}{c^2} - L^2/c^2}\right)^{1/2} = \left|\sqrt{1+x} \sim 1 - \frac{x}{2}\right| =$$

$$= \sqrt{\frac{\omega^2}{c^2} - \frac{L^2}{r^2}} - \frac{\delta c}{c}\,\frac{\omega^2}{c^2}\,\frac{1}{\frac{\omega}{c}\sqrt{1-\frac{L^2c^2}{r^2\omega^2}}} =$$

$$= A - \frac{\delta c}{c}\,\frac{\omega}{c}\,\frac{1}{\sqrt{1-\frac{L^2c^2}{r^2\omega^2}}}$$

comparison:

$$\int_{r_t}^{R_0}\left[\cancel{A} + \frac{\delta\omega}{c}\,\frac{1}{\left(1 - \frac{L^2c^2}{r^2\omega^2}\right)^{1/2}}\right]dr = \int_{r_t}^{R_0}\left[\cancel{A} - \frac{\delta c}{c}\,\frac{\omega}{c}\,\frac{1}{\sqrt{1-\frac{L^2c^2}{r^2\omega^2}}}\right]dr$$

$$\int_{r_t}^{R_0}\frac{\delta\omega}{c}\left(1-\frac{L^2c^2}{r^2\omega^2}\right)^{-1/2}dr = -\int_{r_t}^{R_0}\frac{\delta c}{c}\,\frac{\omega}{c}\left(1-\frac{L^2c^2}{r^2\omega^2}\right)^{-1/2}dr$$

$$\frac{\delta\omega}{\omega}\underbrace{\int_{r_t}^{R_0}\frac{1}{c}\,\frac{1}{\sqrt{1-\frac{L^2c^2}{r^2\omega^2}}}\,dr}_{\text{background travel time}} = -\int_{r_t}^{R_0}\frac{\delta c}{c}\,\frac{1}{c}\,\frac{1}{\sqrt{1-\frac{L^2c^2}{r^2\omega^2}}}\,dr$$

$$\frac{\delta\omega}{\omega} = \underbrace{-\frac{1}{\tau}\int_{r_t}^{R_0}\frac{\delta c}{c}\underbrace{\frac{1}{c}\,\frac{1}{\sqrt{1-\frac{L^2c^2}{r^2\omega^2}}}}_{\text{kernel } K_{\delta c}}dr}_{\text{average perturbation of the speed of sound along propagation ray}}$$

is it a travel time?

~~dt dr w/c 1/k_r~~

propagation path: $\vec{v}_g = \frac{\partial r}{\partial t} = \frac{\partial \omega}{\partial k}$

radial: $\frac{dr}{dt} = \frac{\partial \omega}{\partial k_r}$

angular: $r\frac{d\vartheta}{dt} = \frac{\partial \omega}{\partial k_h}$

dispersion relation: $\omega^2 = c^2(k_r^2 + k_h^2) + \omega_c^2$

for $\omega^2 \gg \omega_c^2$

$$\frac{dr}{dt} = \frac{\partial \omega}{\partial k_r} = \frac{\partial}{\partial k_r}\left(c\sqrt{k_r^2+k_h^2}\right) = \frac{c}{2\sqrt{k_h^2+k_r^2}} \cdot 2k_r =$$

$$= \frac{ck_r}{\omega}$$

$$\Rightarrow \boxed{dt} = dr\,\frac{\omega}{c}\,\frac{1}{k_r} = \frac{\omega}{c^2}\,\frac{1}{\sqrt{\frac{\omega^2}{c^2} - \frac{L^2}{r^2}}}\,dr =$$

$$= dr\,\frac{\omega}{c^2}\,\frac{1}{\frac{\omega}{c}\sqrt{1-\frac{L^2c^2}{\omega^2 r^2}}} = \frac{dr}{c}\,\boxed{\frac{1}{\sqrt{1-\frac{L^2c^2}{r^2\omega^2}}}}$$

$$\tau = \int_{r_t}^{R_\odot} dt = \int_{r_t}^{R_\odot} \frac{1}{c}\,\frac{1}{\sqrt{1-\frac{L^2c^2}{r^2\omega^2}}}\,dr \qquad \text{Q.E.D.}$$

~~... pro~~

equation for the ray:

$$r\frac{\partial\vartheta}{dr} = \frac{\partial\omega}{\partial k_h} \Big/ \frac{\partial\omega}{\partial k_r} = \frac{k_h}{k_r} = \frac{\frac{L}{r}}{\sqrt{\frac{\omega^2}{c^2} - \frac{L^2}{r^2}}}$$

- - - - -

in total

$$\frac{\delta\omega}{\omega} = -\frac{1}{\tau}\int_{r_t}^{R_\odot} \frac{\delta c}{c}\, K_c(l,\omega)\,dr$$

possible to invert analytically for c

= XVII =

generalisation

another observable – travel time

$$\tau = \int_r^{R_\odot} \frac{1}{c} \frac{dr}{\sqrt{1-\frac{L^2 c^2}{r^2 \omega^2}}} \;; \quad c = c(q_\alpha)$$

$$q_\alpha \equiv \{ \rho, \delta, \vec{v}, \mathcal{B}, \dots \}$$

travel-time measurements by cross-correlation

$\phi(k, \omega)$ ... measurement

$F_a(k, \omega)$ ... filter

$$\psi(k, \omega) = F_a(k, \omega)\, \phi(k, \omega)$$

$$c(x_1, x_2, t) = \frac{h_t}{T - |t|} \sum_{t'} \psi(x_1, t')\, \psi(x_2, t'+t) \sim$$

$$\sim \int_0^T dt\; \psi(x_1, t')\, \psi(x_2, t'+t)$$

fitem τ_a^0 ... pozorovaný travel-time, kdy $c(x_1, x_2, t)$ maximální!

pak:

$$\chi = \frac{1}{2} \sum_a \underbrace{(\tau_a - \tau_a^0)^2}_{\delta\tau}$$

misfit between observations and a model

$$\delta\chi = \sum_a (\tau_a - \tau_a^0)^2 \underbrace{\frac{\partial \tau_a}{\partial q_\alpha}} \delta q_\alpha$$

$K_\alpha^a(x, q_\alpha)$ ... sensitivity kernel

Fréchet derivative

generally

formal expression: $\omega^2 \vec{q_\alpha} = \mathcal{L}(\vec{q_\alpha})$ $\quad q_\alpha$ ... parameters

↳ frequencies depend on perturbations $\Rightarrow$ eigenproblem

we multiply by q_α^* and integrate over the Sun:

$$\omega^2 \int_\odot q_\alpha^* \cdot q_\alpha \, d^3r = \int_\odot q_\alpha^* \cdot \mathcal{L}(q_\alpha)\, d^3r$$

$$\omega^2 = \frac{\int_\odot q_\alpha^* \, \mathcal{L}(q_\alpha)\, d^3r}{\int_\odot q_\alpha^* \cdot q_\alpha \, d^3r}$$

solar parameters in $\mathcal{L}$ (internal structure), frequencies are eigenvalues of q_α

model perturbation $\rightarrow$ frequency shifts

small, $\mathcal{L}(q_\alpha) = \mathcal{L}_0(q_\alpha) + \mathcal{L}_1(q_\alpha)$

$$\Rightarrow \omega^2 + \delta\omega^2 = \frac{\int_\odot d^3r \, q_\alpha^* \cdot \mathcal{L}(q_\alpha)}{\int_\odot q_\alpha^* \cdot q_\alpha \, d^3r} =$$

$$= \frac{\int_\odot d^3r \, q_\alpha^* \cdot \mathcal{L}_0(q_\alpha)}{\int_\odot d^3r \, q_\alpha^* \cdot q_\alpha} + \frac{\int_\odot d^3r \, q_\alpha^* \, \mathcal{L}_1(q_\alpha)}{\int_\odot q_\alpha^* \cdot q_\alpha \, d^3r}$$

$$\Rightarrow \delta\omega^2 = \frac{\int_\odot d^3r \, q_\alpha^* \, \mathcal{L}_1(q_\alpha)}{\int_\odot q_\alpha^* \, q_\alpha \, d^3r} \leftarrow \text{mode mass } I$$

$$\delta\omega^2 \sim 2\omega_0 \, \delta\omega = \frac{\int_\odot d^3r \, q_\alpha^* \, \mathcal{L}_1(q_\alpha)}{I} \quad / : 2\omega_0^2$$

$$\Rightarrow \frac{\delta\omega}{\omega} = \frac{\int_\odot d^3r \, q_\alpha^* \cdot \mathcal{L}_1(q_\alpha)}{2\omega_0^2 I}$$

$\mathcal{L}_1$ needs to be given: operator splitting

$$\Rightarrow \frac{\delta\omega}{\omega} = \int_\odot d^3r \Big[\underbrace{K_\rho \frac{\delta\rho}{\rho} + K_{c^2} \frac{\delta c^2}{c^2}}_{\text{sensitivity kernels}} + K_{\vec{v}} \cdot \delta\vec{v} + \dots \Big]$$

$K_{\vec{v}} \cdot \delta\vec{v} \rightarrow K_{v_x} \delta v_x + K_{v_y} \delta v_y + K_{v_z} \delta v_z +$

How to calculate the kernels?

formal solution, pairs of variables: $\vec{X} = \left(\frac{\delta\rho}{\rho}, \frac{\delta r}{r}\right)$; $\vec{Y} = \left(\frac{\delta\omega}{\omega}, \frac{\delta c}{c}\right)$

linearised structural equations bind those pairs

$$A\vec{X} = \vec{Y}$$

if $\vec{K}_X$ & $\vec{K}_Y$ kernels for X & Y then:

$$\frac{\delta\omega}{\omega} = \int_\odot K_X \cdot X \, d^3r \equiv \langle K_X \cdot X \rangle$$

and

$$\frac{\delta\omega}{\omega} = \langle K_Y \cdot Y \rangle \quad \text{applying } AX = Y:$$

$$\langle K_Y \cdot Y \rangle = \langle K_Y \cdot AX \rangle = \langle A^* K_Y \cdot X \rangle \overset{!}{\equiv} \langle K_X \cdot X \rangle$$

$$\Rightarrow \langle A^* K_Y \cdot X \rangle = \langle K_X \cdot X \rangle$$

$$\Rightarrow A^* \vec{K}_Y = \vec{K}_X$$

equations for kernel are adjugated to equations of internal structure

$$\boxed{\text{Time - distance}}$$

forward:

$$\delta\tau^a(r) = \int_\odot d^2r \, dz \sum_{\beta=1}^{P} K^a_\beta(r - r, z) \, \delta q_\alpha(r, z) + n^a(r)$$

noise covariance matrix:

$$\Lambda_{ab}(r_i - r_j) = \mathrm{cov}[n^a(r_i) \; n^b(r_j)]$$

$$\mathrm{Tr}[\Lambda_{ab}] = \sigma_a^2$$

inversion:

$\boxed{\text{RLS}}$ - regularised least squares:

$$\chi^2 = \sum_a \frac{1}{\sigma_a^2} \left[\delta\tau^a - \int_\odot d^2r \, dz \sum K^a_\beta \, \delta q_\beta \right]^2 + \mu L(q_\alpha)$$

operator of the regularisation

$$\text{e.g. } L = |q_\alpha|^2 \quad \text{or} \quad L = \left(\frac{\partial^2 q_\alpha}{\partial z^2} \right)^2$$

after minimisation: $\frac{\partial \chi^2}{\partial q_\alpha}$ ... system of equations to be solved

= XX =

$\boxed{\text{OLA}}$ – optimally localised averaging

seeking a solution: forward problem

$$\delta q_\alpha^{inv}(r_0, z_0) = \sum_i \sum_a w_a^\alpha(r_i - r_0, z_0)\, \delta\tau^a(r_i) =$$

$$= \int_\odot d^2r'\,dz \sum_\beta \Big[\sum_{i,a} w_a^\alpha(r_i - r_0; z_0)\, K_\beta^a(r_i' - r_i, z) \Big] \delta q_\beta(r', z) +$$

$$+ \sum_{i,a} w_a^\alpha(r_i - r_0; z_0)\, u^a(r_i)$$

definition: $\mathcal{K}_\beta^\alpha(r, z; z_0) = \sum_i \sum_a w_a^\alpha(r_i, z_0)\, K_\beta^a(r - r_i, z)$ $\forall\, \beta \in P$

$\hookrightarrow$ averaging kernel

then:

$$\delta q_\alpha^{inv}(r_0, z_0) = \int_\odot d^2r'\,dz\, \mathcal{K}_\alpha^\alpha(r' - r_0; z; z_0)\, \delta q_z(r', z) +$$

$$+ \int_\odot d^2r'\,dz \sum_{\beta, \beta \neq \alpha} \mathcal{K}_\beta^\alpha(r' - r_0; z; z_0)\, \delta q_\beta(r', z) +$$

$$+ \sum_{i,a} w_a^\alpha(r_i - r_0; z_0)\, u^a(r_i)$$

cost function

$$\chi_\alpha^2(w^\alpha, \mu) = \int_\odot d^2r\,dz \sum_\beta \left[\mathcal{K}_\beta^\alpha - T_\beta^\alpha\right]^2 + \mu \sum_{i,j,a,b} w_a^\alpha \Lambda_{ab} w_b^\alpha$$

solution for ~~return pro~~ $\frac{\partial \chi_\alpha}{\partial w^\alpha} \stackrel{!}{=} 0$

RMS ~~of~~ of the random noise

$$\sigma_\alpha^2 = \sum_{a,b,i,j} w_a^\alpha(r_i)\, \Lambda_{ab}(r_i - r_j)\, w_b^\alpha(r_j)$$

+ constraint $\int_\odot d^2r\,dz\, \mathcal{K}_\beta^\alpha = \delta_{\alpha\beta}$

joined using ~~an~~ Lagrange multipliers