Oscillations

→ in Doppler-shifts, intensity, ... ← measurements
in Cartesian system → Fourier components usefull

$$a(k_x, k_y, \omega) = \iiint dx\, dy\, dt \; v(x, y, t)\, e^{i(k_x x + k_y y + \omega t)}$$

$$(k_x, k_y) = \vec{k}_h \; ; \quad |k| = \frac{2\pi}{\lambda}$$

power spectrum $P(k_x, k_y, \omega) = a^* a$

on a sphere → spherical functions

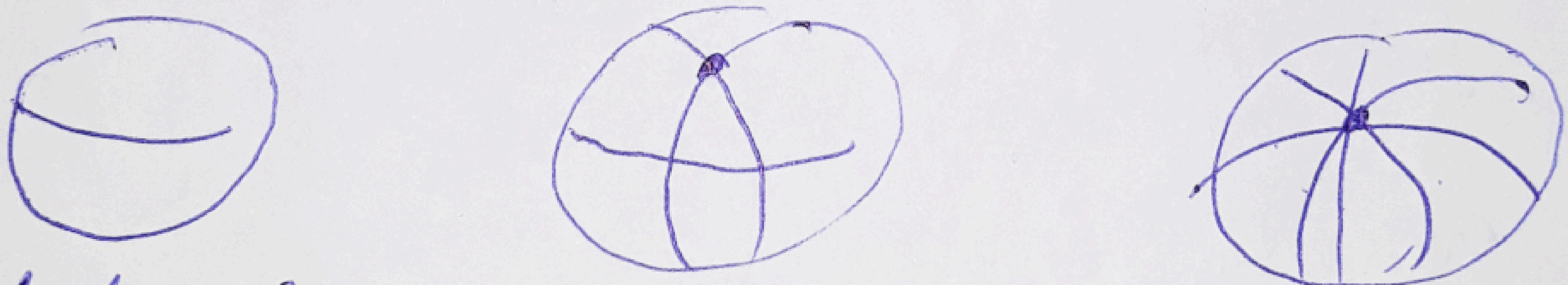
$$v = v(\vartheta, \varphi, t)$$

$$a(l, m, \omega) = \iiint v(\vartheta, \varphi, t) \underbrace{Y_l^m}_{\text{spherical function}} (\vartheta, \varphi)\, e^{i\omega t} \, d\vartheta\, d\varphi\, dt$$

power spectrum: $P(l, m, \omega) = a^* a = |a^2|$

in a spherical symmetry $P \neq P(m)$

↑ degeneration, possibly removed by e.g. rotation



$l = 1, m = 0$ $\qquad$ $l = 3, m = 2$ $\qquad$ $l = 4, m = 4$

Measurements: theorems of DFT

signal measured over time T

⇒ frequency resolution $\Delta\omega = \frac{2\pi}{T}$

↑ also the lowest measurable frequency

highest measurable frequency → Nyquist frequency

$$\omega_{Ny} = \frac{\pi}{\Delta t}$$ Δt sampling in time

larger frequencies than → sample to the lower ones = <u>aliasing</u>

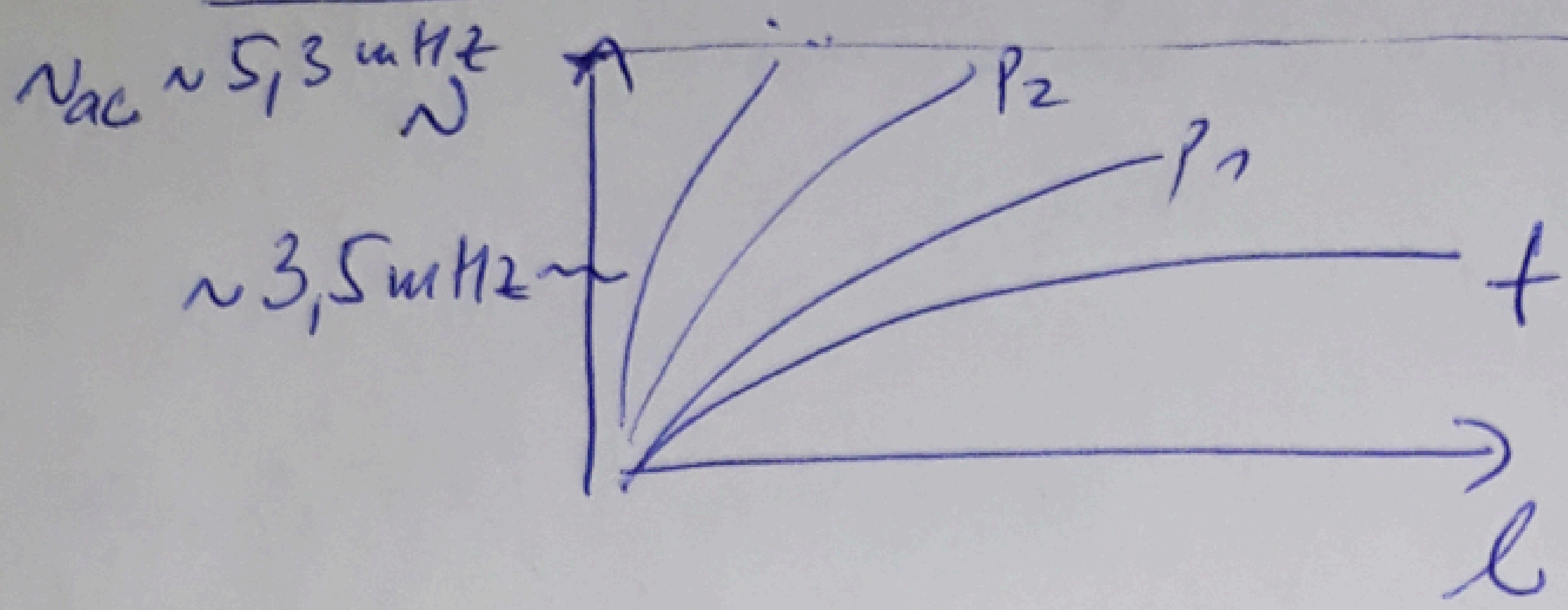
Hence: $\Delta\omega = \frac{2\pi}{T} \leq \omega \leq \frac{\pi}{\Delta t}$

spatial frequencies / resolutions by analogy

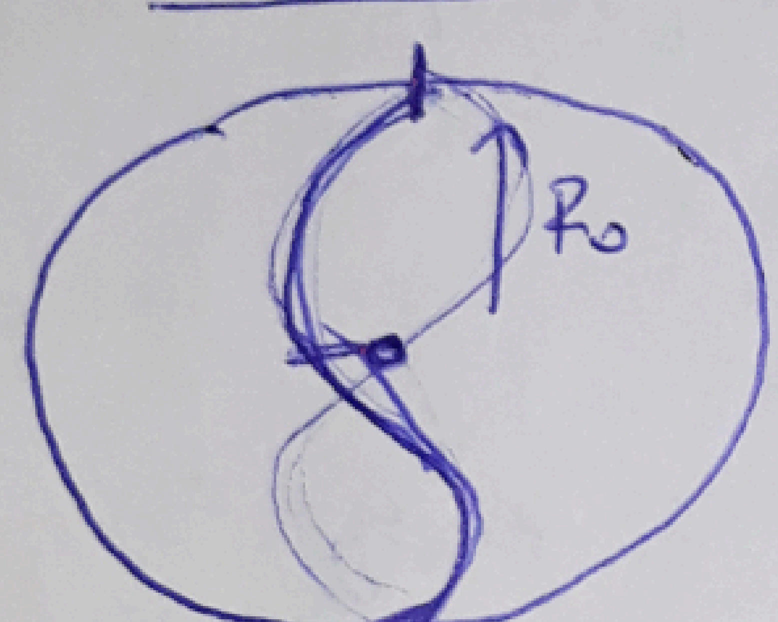
<u>Problem:</u> observations of a hemisphere only

$\rightarrow Y_l^m$ not orthogonal $\Rightarrow$ aliasing, false modes

<u>solar oscillations spectrum</u>



$\boxed{\text{Fundamental mode}}$



$\lambda = 4R_\odot$ $\rightarrow$ a wave back and forth travelling with speed of sound

$$c_s = \sqrt{\frac{\gamma \bar{P}}{\bar{\rho}}}$$ mean quantities

$\bar{P}$ — from eqs. of internal structure

$$\bar{\rho} = \frac{3}{4\pi} \frac{M_\odot}{R_\odot^3}$$

$$\frac{dr}{dm} = \frac{1}{4\pi\rho r^2} \; ; \quad \frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$$

two-point discretisation:

$$\frac{R_\odot - 0}{M_\odot - 0} = \frac{1}{4\pi \bar{\rho} R_\odot^2} \; ; \quad \frac{0 - \bar{P}}{M_\odot - 0} = -\frac{GM_\odot}{4\pi R_\odot^4}$$

$$\bar{P} = \frac{G}{4\pi R_\odot^2} \left(\frac{M_\odot}{R_\odot}\right)^2$$

$$c_s = \left[\gamma \frac{\bar{P}}{\bar{\rho}}\right]^{+1/2} = \left[\gamma \frac{\frac{G}{4\pi R_\odot^2} \frac{M_\odot^2}{R_\odot^2}}{\frac{3}{4\pi} \frac{M_\odot}{R_\odot^3}}\right]^{1/2} = \left[\gamma \frac{G}{3} \frac{M_\odot}{R_\odot}\right]^{1/2}$$

oscillation: forth and back

$$\tau = \frac{\lambda}{c_s} = \frac{4R_\odot}{c_s} = \left[\frac{16R_\odot^2}{\gamma\,\frac{6}{3}\,\frac{M_\odot}{R_\odot}}\right]^{1/2} = \left[\frac{16}{3\gamma G}\,\frac{R_\odot^3}{M_\odot}\right]^{1/2} =$$

$$= \left[\frac{16}{3\gamma G}\,\frac{3}{4\pi}\,\frac{\frac{4\pi}{3}R_\odot^3}{M_\odot}\right]^{1/2} = \sqrt{\frac{4}{G\gamma\pi}\,\frac{1}{\bar{\rho}}} = \frac{2}{\sqrt{\gamma\pi}}\left[G\bar{\rho}\right]^{-1/2}$$

for the $\odot$: $G = 6{,}67 \times 10^{-11}\ m^3 kg^{-1} s^{-2}$

$\bar{\rho} = 1{,}409 \times 10^3\ kg/m^3$

$\Rightarrow \tau = 47\ min$

Linear adiabatic oscillations of the non-rotating sun

assumptions: linear: $\frac{\bar{v}}{c_s} \ll 1$

adiabatic: $\frac{ds}{dt} = 0$

spherically symmetric background

no magnetism, Reynolds stress negligible

equations: continuity: $\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho v) = 0$

motion: $\rho\frac{dv}{dt} = -\nabla p - \rho g \quad ; g = \nabla\phi$

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + (v\cdot\nabla)v$$

adiabaticity: $pV^\gamma = \frac{p}{\rho^\gamma} = const$

$$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0$$

$$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = \frac{dp}{dt}\frac{1}{\rho^\gamma} + p\frac{d}{dt}\frac{1}{\rho^\gamma} = \left(\frac{dp}{dt} - p\gamma\frac{1}{\rho}\frac{d\rho}{dt}\right)\frac{1}{\rho^\gamma} = 0$$

$$\frac{dp}{dt} - \underbrace{\frac{p\gamma}{\rho}}_{c_s^2}\frac{d\rho}{dt} = \frac{dp}{dt} - c_s^2\frac{d\rho}{dt} = 0$$

Poisson: $\Delta\phi = 4\pi G\rho$

=III=

perturbation: $v_0 = 0$, $\rho_0 = \rho_0(r)$, $p_0 = p_0(r)$

$\vec{\xi}(t)$ ... displacement

$$\vec{v} = \frac{d\vec{\xi}}{dt}$$

perturbation: $p, \rho, v \rightarrow$

Eulerian → at a position

$$\rho(r,t) = \rho_0(r) + \rho'(r,t)$$

Lagrangian → on a particle

$$\delta\rho(r+\xi) = \rho_0(r) + \delta\rho(r,t)$$

Lagrange vs. Euler — only radial component

$$\delta\rho = \rho' + (\xi \cdot \nabla\rho_0) = \rho' + (\xi \cdot e_r)\frac{d\rho_0}{dr} =$$

$$= \rho_0 + \xi_r \frac{d\rho_0}{dr}$$

→ radial displacement

<u>Linearisation</u> $\rho = \rho_0 + \rho'$, $v = v_0 + v'$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho v) = 0; \quad \rho' v' \rightarrow 0, \quad \nabla\cdot v_0 = 0, \quad v_0 = 0$$

$$\underbrace{\frac{\partial\rho_0}{\partial t} + \nabla\cdot(\rho_0 v_0)}_{=0,\ \text{background solution satisfied}} + \frac{\partial\rho'}{\partial t} + \nabla\cdot(\rho_0 v') = 0$$

$$\frac{\partial\rho'}{\partial t} + \nabla\cdot(\rho_0 v') = 0 \quad ; \quad v' = \frac{\partial\xi'}{\partial t}$$

$$\frac{\partial\rho'}{\partial t} + \nabla\cdot\left(\rho_0 \frac{\partial\xi}{\partial t}\right) = 0$$

$$\frac{\partial\rho'}{\partial t} + \frac{\partial}{\partial t}\nabla\cdot(\rho_0\xi) - \nabla\cdot\left(\xi\frac{\partial\rho_0}{\partial t}\right) = 0$$

$$\frac{\partial\rho'}{\partial t} + \frac{\partial}{\partial t}\nabla\cdot(\rho_0\xi) = \nabla\cdot\left(\xi\frac{\partial\rho_0}{\partial t}\right) = -\nabla\cdot\left[\xi\nabla\cdot(\rho_0 v_0)\right]$$

$$\frac{\partial\rho_0}{\partial t} + \nabla\cdot(\rho_0 v_0) = 0$$

integrate

$$\rho' + \nabla\cdot(\rho_0\xi) = -\int \underbrace{\nabla\cdot\left[\xi\nabla\cdot(\rho_0 v_0)\right]}_{=0} dt = 0$$

for each t

- N =

hence linearised set

$$\rho' + \nabla\cdot(\rho_0 \vec{\xi}) = 0 \qquad (1)$$

$$\rho_0 \frac{\partial v'}{\partial t} = -\nabla p' - g e_r \rho' + \rho_0 \nabla \theta' \qquad (2)$$

$$p' + \xi_r \frac{dp_0}{dr} = c_0^2 \left(\rho' + \xi_r \frac{d\rho_0}{dr}\right) \qquad (3)$$

$$\Delta \theta' = 4\pi G \rho' \qquad (4)$$

+ Cowling approximation - waves small amplitude → they can't possibly influence a gravity potential

$$\Rightarrow \theta' = 0$$

+ spherical geometry (r, ϑ, φ)

$$\vec{\xi} = \xi_r e_r + \vec{\xi}_h \leftarrow \text{stratification radial, hence } \vec{\xi}_h \text{ symmetrical}$$

$$\nabla\cdot\vec{\xi} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \xi_r) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta\, \xi_\vartheta) + \frac{1}{r\sin\vartheta}\frac{\partial \xi_\varphi}{\partial\varphi} =$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\xi_r) + \frac{1}{r}\nabla_h\cdot\vec{\xi}_h$$

(1) $\rho' + \nabla\cdot(\rho_0\vec{\xi}) = 0$

$$\rho' + \nabla_r\cdot(\rho_0 \vec{\xi}_r) + \nabla_h\cdot(\rho_0 \xi_h) = 0$$

$$\rho' + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho_0\xi_r) + \frac{\rho_0}{r}\nabla_h\cdot\vec{\xi}_h = 0$$

(2) $\rho_0 \frac{\partial v'}{\partial t} = -\nabla p' - g e_r \rho' + 0 \leftarrow$ Cowling

$$v' = \frac{\partial \vec{\xi}}{\partial t}$$

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = -\nabla p' - g e_r \rho'$$

$$\rho_0\left[\frac{\partial^2\xi_r}{\partial t^2} + \frac{\partial^2 \xi_h}{\partial t^2}\right] = -\nabla_r p' - \frac{1}{r}\nabla_h p' - g e_r \rho'$$

split into two equations:

$$\rho_0 \frac{\partial^2\xi_r}{\partial t^2} = -\nabla_r p' - g e_r \rho'$$

$$\rho_0 \frac{\partial^2\xi_h}{\partial t^2} = -\frac{1}{r}\nabla_h p'$$

$\Rightarrow$ V $\Rightarrow$

seeking for solution by waves: $\xi_r \sim e^{i\omega t}$, $\xi_h \sim e^{i\omega t}$

$$\Rightarrow -\omega^2 \rho_0 \xi_r = -\frac{\partial p'}{\partial r} - g\rho'$$

$$-\omega^2 \rho_0 \xi_h = -\frac{1}{r}\nabla_h p'$$

(3) $\frac{dp}{dt} = c^2 \frac{d\rho}{dt} \Rightarrow \delta p = c^2 \delta\rho$

$$\delta p = p' + \xi_r \frac{dp_0}{dr}$$

$$\delta\rho = \rho' + \xi_r \frac{d\rho_0}{dr}$$

$$p' + \xi_r \frac{dp_0}{dr} = c_0^2\left(\rho' + \xi_r \frac{d\rho_0}{dr}\right)$$

$$\frac{1}{c_0^2}p' + \frac{1}{c_0^2}\xi_r \frac{dp_0}{dt} = \rho' + \xi_r \frac{d\rho_0}{dr}$$

$$\rho' = \frac{1}{c_0^2}p' + \frac{1}{c_0^2}\xi_r \frac{dp_0}{dr} - \xi_r \frac{d\rho_0}{dr} =$$

$$= \frac{1}{c_0^2}p' + \xi_r\left[\frac{1}{c_0^2}\frac{dp_0}{dr} - \frac{d\rho_0}{dr}\right] =$$

$$= \frac{1}{c_0^2}p' + \xi_r\left[\frac{\rho_0}{\gamma p_0}\frac{dp_0}{dr} - \frac{d\rho_0}{dr}\right] =$$

$$= \frac{1}{c_0^2}p' + \xi_r \frac{\rho_0}{g} \underbrace{g\left[\frac{1}{\gamma p_0}\frac{dp_0}{dr} - \frac{1}{\rho_0}\cdot\frac{d\rho_0}{dr}\right]}_{\text{Brunt-Väisälä}} =$$

$$= \frac{1}{c_0^2}p' + \xi_r \frac{\rho_0}{g}N^2 =$$

$$\Rightarrow \rho' = \frac{p'}{c_0^2} + \frac{\rho_0 N^2}{g}\xi_r$$

& N^2 from mixing-length:

$$N^2 = \frac{g}{\rho}\left[\left(\frac{d\rho}{dr}\right)_{ad} - \frac{d\rho}{dr}\right];$$ $P\rho^{-\gamma} = \text{const}$ differentiate

$$\Rightarrow dP\rho^{-\gamma} - \gamma P\rho^{-\gamma-1}d\rho = 0$$

$$dP\rho^{-\gamma} = \gamma P\rho^{-\gamma-1}d\rho \Rightarrow dp = \gamma p\rho^{-1}d\rho$$

$$N^2 = \frac{g}{\rho}\left[\frac{\rho}{\gamma p}\frac{dp}{dr} - \frac{d\rho}{dr}\right] = \boxed{g\left[\frac{1}{\gamma p}\frac{dP}{dr} - \frac{1}{\rho}\frac{d\rho}{dr}\right]}$$

it is the same

(4) neglected

→ new 4 equations:

$$\rho' + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho_0 \xi_r) + \frac{\rho_0}{r}\nabla_h \cdot \xi_h = 0 \qquad (a)$$

$$-\omega^2 \rho_0 \xi_r = -\frac{\partial p'}{\partial r} - g\rho' \qquad (b)$$

$$-\omega^2 \rho_0 \xi_h = -\frac{1}{r}\nabla_h p' \qquad (c)$$

$$\rho' = \frac{p'}{c_0^2} + \frac{\rho_0 N^2}{g}\xi_r \qquad (d)$$

boundary conditions: $\xi_r(r=0) = 0$ (regularity for $l=1$) center stable!

$\delta p(r = R_\odot) = 0$ → no external forces

for $\vartheta = 0, \pi$ solution regular

⇒ separate the radial and angular parts

$$\rho'(r,\vartheta,\varphi) = \rho'(r)\cdot f(\vartheta,\varphi)$$
$$p'(r,\vartheta,\varphi) = p'(r)\cdot f(\vartheta,\varphi)$$
$$\xi_r(r,\vartheta,\varphi) = \xi_r(r)\, f(\vartheta,\varphi)$$
$$\xi_h(r,\vartheta,\varphi) = \xi_h(r)\cdot \nabla_h f(\vartheta,\varphi)$$

(a) $$\left[\rho' + \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \rho_0 \xi_r\right] f(\vartheta,\varphi) + \frac{\rho_0}{r}\xi_h \nabla_h^2 f = 0$$

separate, $\nabla_h^2 f = \alpha f$, $\alpha =$ const ⇝ Laplace equation

nontrivial solution

$= \alpha = -l(l+1)$

$$\Rightarrow \underline{f(\vartheta,\varphi)} = \underline{\underline{Y_l^m(\vartheta,\varphi)}} = C\, \underbrace{P_l^m(\vartheta)}_{\text{Legendre}}\, e^{im\varphi}$$

$$\Rightarrow \rho' + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho_0\xi_r) - \frac{l(l+1)}{r}\rho_0 \xi_h = 0$$

(c) $$-\omega^2 \rho_0 \xi_h(r)\cdot \cancel{\nabla_h f(\vartheta,\varphi)} = -\frac{1}{r}p'(r)\, \cancel{\nabla_h f(\vartheta,\varphi)}$$

$$-\omega^2 \rho_0 \xi_h(r) = -\frac{1}{r}p'(r)$$

$\xi_h(r) = \frac{1}{\omega^2 \rho_0 r} p'$ (*) put in the continuity equation:

$$\rho' + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho_0 \xi_r) - \frac{l(l+1)}{r}\rho_0 \xi_h = 0$$

$$\rho' + \frac{2}{r}\rho_0 \xi_r + \xi_r \frac{d\rho_0}{dr} + \rho_0 \frac{d\xi_r}{dr} - \frac{l(l+1)}{\omega^2 r^2} p' = 0$$

$\hookrightarrow$ from (d) $\rho' = \frac{p'}{c_0^2} + \frac{\rho_0 N^2}{g}\xi_r$

$$\rho_0 \frac{d\xi_r}{dr} + \xi_r \frac{d\rho_0}{dr} + \frac{2}{r}\rho_0 \xi_r + \frac{p'}{c_0^2} + \frac{\rho_0 N^2}{g}\xi_r - \frac{l(l+1)}{\omega^2 r^2} p' = 0 \Big/ : \rho_0$$

$$\frac{d\xi_r}{dr} + \frac{2}{r}\xi_r + \xi_r\left[\frac{1}{\rho_0}\frac{d\rho_0}{dr} + \frac{N^2}{g}\right] + \frac{p'}{\rho_0 c_0^2}\left[1 - \frac{l(l+1)c_0^2}{r^2\omega^2}\right] = 0$$

$$\frac{1}{\rho_0}\frac{d\rho_0}{dr} + \frac{N^2}{g} = \cancel{\frac{1}{\rho_0}\frac{d\rho_0}{dr}} + \frac{1}{\gamma p_0}\frac{dp_0}{dr} - \cancel{\frac{1}{\rho_0}\frac{d\rho_0}{dr}} = \frac{1}{\gamma p_0}\frac{dp_0}{dr} =$$

$$= \left| \frac{dp_0}{dr} = -g\rho_0 \text{ hydrostat. equilibrium} \right| = -\frac{1}{\gamma p_0} g \rho_0 = \left| c_0^2 = \frac{\gamma p_0}{\rho_0} \right| =$$

$$= -\frac{g}{c_0^2}$$

hence

$$\frac{d\xi_r}{dr} + \cancel{\frac{2}{r}\xi_r} - \frac{g}{c_0^2}\xi_r + \left[1 - \frac{l(l+1)c_0^2}{r^2\omega^2}\right]\frac{p'}{\rho_0 c_0^2} = 0 \quad (1)$$

$\hookrightarrow$ dimension of frequency

$\frac{d\xi_r}{dr} \gg \frac{\xi_r}{r}$; $\xi_r \ll r$

$\Rightarrow S_l^2 = \frac{l(l+1)c^2}{r^2}$

$\hookrightarrow$ Lamb frequency

(b) $-\omega^2 \rho_0 \xi_r = -\frac{dp'}{dr} - g\rho'$

$$\frac{dp'}{dr} + g\rho' - \omega^2 \rho_0 \xi_r = 0$$

$\hookrightarrow \rho' = \frac{p'}{c_0^2} + \frac{\rho_0 N^2}{g}\xi_r$

$$\Rightarrow \frac{dp'}{dr} + \frac{g}{c_0^2}p' + (N^2 - \omega^2)\rho_0 \xi_r = 0 \quad (2)$$

boundary conditions:

down $\xi_r = 0$ (3)

up $\delta p = p' + \frac{dp_0}{dr}\xi_r = 0 \Rightarrow \frac{dp_0}{dr} = -g\rho_0$ (4)

testing (4): $p' - g\rho_0 \xi_r = 0$

$\xi_h = \frac{1}{\omega^2 \rho_0 r} p' \rightarrow \omega^2 \rho_0 r \xi_h - g \rho_0 \xi_r = 0$

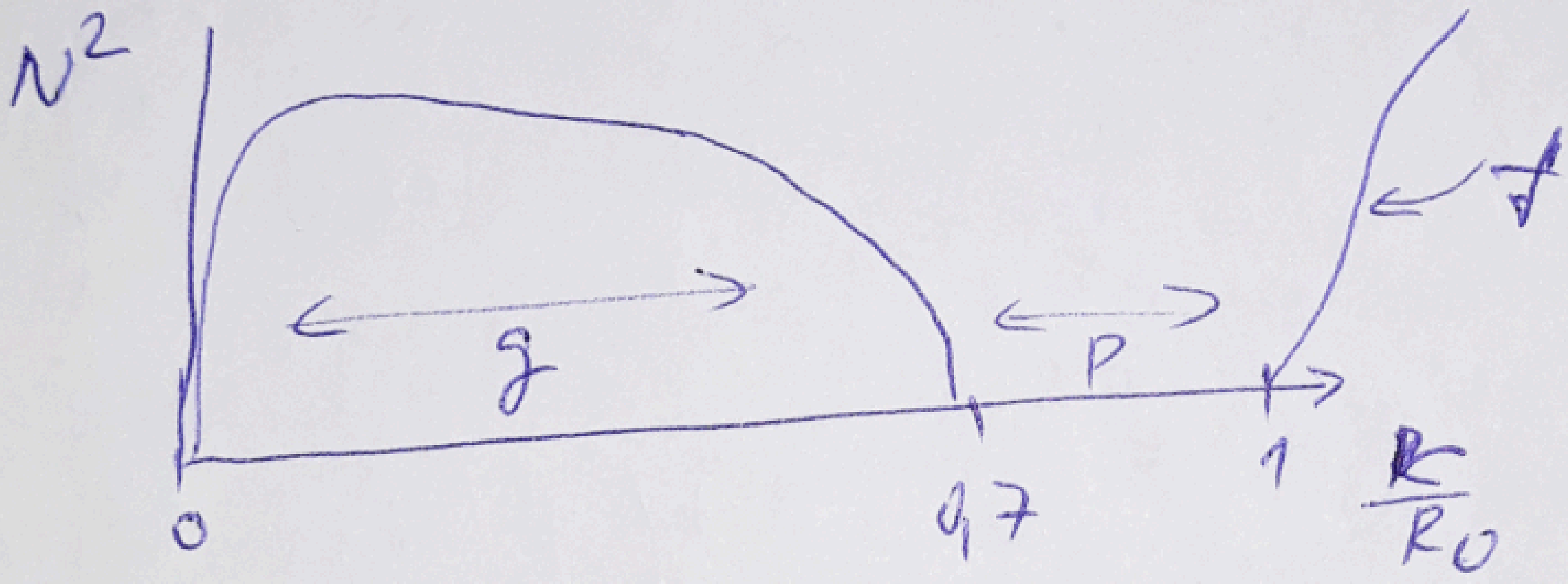
$\Rightarrow \frac{\xi_h}{\xi_r} = \frac{g}{\omega^2 r}$ should be valid

↳ not exactly in reality, deviations caused by the existence of an atmosphere

in total: eqs (1) & (2) + boundary conditions (3) & (4)

→ eigenvalue problem for oscillations

three modes observed



$\boxed{\text{JWKB solution}}$

Jeffreys–Wentzel–Kramers–Brillouin

assumption: within the wave mostly density is changed

seeking solution: $\xi_r = A\,\rho(r)^{-1/2} e^{-ik_r r}$

$p' = B\,\rho(r)^{1/2} e^{-ik_r r}$

$k_r = k_r(r)$ slowly changing

(1) $\frac{d\xi_r}{dr} - \frac{g}{c^2}\xi_r + \left[1 - \frac{S_l^2}{\omega^2}\right]\frac{p'}{\rho_0 c_0^2} = 0$

$A\left[-ik_r \rho_0^{-1/2} e^{-ik_r r} - \frac{1}{2}\rho_0^{-3/2} e^{-ik_r r}\frac{d\rho}{dr}\right] - \frac{g}{c^2} A \rho_0^{-1/2} e^{-ik_r r} +$

$+ \left[1 - \frac{S_l^2}{\omega^2}\right]\frac{B \rho_0^{1/2}}{\rho_0 c_0^2} e^{-ik_r r} = 0$

$$A\left[-ik_r \rho^{-1/2} - \frac{1}{2}\frac{1}{\rho_0}\rho_0^{-1/2}\frac{\partial \rho_0}{\partial r}\right] - \frac{g}{c^2}A\rho^{-1/2} + \left[1 - \frac{S_\ell^2}{\omega^2}\right]\frac{B\rho^{-1/2}}{c^2} = 0 \quad / : \rho^{-1/2}$$

$$\Big[-ik_r - \frac{1}{2}\underbrace{\frac{1}{\rho}\frac{\partial \rho}{\partial r}}_{-H_\rho^{-1}} - \frac{g}{c^2}\Big]A + \left[1 - \frac{S_\ell^2}{\omega^2}\right]\frac{B}{c^2} = 0$$

$$\left[-ik_r + \frac{1}{2H_\rho} - \frac{g}{c^2}\right]A + \frac{1}{c^2}\left[1 - \frac{S_\ell^2}{\omega^2}\right]B = 0$$

② $\frac{\partial p'}{\partial r} + \frac{g}{c^2}p' + (N^2 - \omega^2)\rho_0 \xi_r = 0$

$$Be^{-ik_r r}\left[-ik_r \rho^{1/2} + \frac{1}{2}\rho^{-1/2}\frac{\partial \rho}{\partial r}\right] + \frac{g}{c^2}B\rho^{1/2}e^{-ik_r r} + (N^2 - \omega^2)\rho A\rho^{-1/2}e^{-ik_r r} = 0 \quad / : \rho^{1/2}$$

$$\left[-ik_r + \frac{1}{2}\frac{1}{\rho}\frac{\partial \rho}{\partial r}\right]B + \frac{g}{c^2}B + (N^2 - \omega^2)A = 0$$

$$\left[-ik_r - \frac{1}{2H_\rho}\right]B + \frac{g}{c^2}B + (N^2 - \omega^2)A = 0$$

equations

$$A\left[-ik_r + \frac{1}{2H} - \frac{g}{c^2}\right] + B\left[\frac{1}{c^2} - \frac{S_\ell^2}{\omega^2 c^2}\right] = 0$$

$$A\left[N^2 - \omega^2\right] + B\left[-ik_r - \frac{1}{2H} + \frac{g}{c^2}\right] = 0$$

$\det[\ ] = 0$

$$\left[-ik_r + \left(\frac{1}{2H} - \frac{g}{c^2}\right)\right]\left[-ik_r - \left(\frac{1}{2H} - \frac{g}{c^2}\right)\right] - \left(\frac{1}{c^2} - \frac{S_\ell^2}{\omega^2 c^2}\right)(N^2 - \omega^2) = 0$$

$$-k_r^2 - \left(\frac{1}{2H} - \frac{g}{c^2}\right)^2 - \frac{1}{c^2}(N^2 - \omega^2) + \frac{S_\ell^2}{\omega^2 c^2}(N^2 - \omega^2) = 0$$

$$k_r^2 = \frac{1}{4H^2} - \frac{2}{2H}\frac{g}{c^2} + \frac{g^2}{c^4} - \frac{N^2}{c^2} + \frac{\omega^2}{c^2} + \frac{S_\ell^2}{\omega^2 c^2}(N^2 - \omega^2) =$$

$$= \frac{\omega^2 - \overbrace{\frac{c^2}{4H^2}}^{\omega_c^2} - N^2 - \frac{1}{H}g + g^2/c^2}{c^2} + \frac{S_\ell^2}{\omega^2 c^2}(N^2 - \omega^2) =$$

$$= \frac{\omega^2 - \omega_c^2 - N^2 + \left(\frac{g^2}{c^2} - \frac{g}{H}\right)}{c^2} + \frac{S_\ell^2}{\omega^2 c^2}(N^2 - \omega^2) =$$

$$N^2 = g\left(\frac{1}{\Gamma_1 p}\frac{\partial p}{\partial r} - \frac{1}{\rho}\frac{\partial \rho}{\partial r}\right) = g\left(\frac{\rho}{\Gamma_1 p}\frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{1}{H}\right) = g\left(\frac{1}{c^2}\frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{1}{H}\right) =$$

$$= \left|\frac{\partial p}{\partial r} = \rho g\right| = g\left(\frac{1}{c^2}\frac{1}{\rho}\rho g - \frac{1}{H}\right) = g\left(\frac{g}{c^2} - \frac{1}{H}\right) = \frac{g^2}{c^2} - \frac{g}{H}$$

$= X =$

$$= \boxed{\frac{\omega^2-\omega_c^2}{c^2} + \frac{S_l^2}{\omega^2 c^2}(N^2-\omega^2) = k_r^2}$$ dispersion relation

ω_c .. acoustic cut-off

for $k_r > 0$ ... propagation, for $k_r < 0$ ... attenuation

resonance: two return points r_1, r_2

$$\int_{r_1}^{r_2} k_r \, dr = \pi(n+\alpha)$$

: n ... order radial

α ... phase change, depends on the properties of the boundary

Approximations

1. $\omega^2 >> N^2$

$$k_r^2 = \frac{\omega^2-\omega_c^2}{c^2} - \frac{S_l^2}{c^2}$$

$$k_h = \frac{S_l}{c} = \frac{\sqrt{l(l+1)}}{r}$$

$$\Rightarrow k_r^2 c^2 = \omega^2 - \omega_c^2 - k_h^2 c^2$$

$$\omega^2 = \omega_c^2 + k_r^2 c^2 + k_h^2 c_s^2 \quad ; \quad k^2 = k_r^2 + k_h^2$$

$$\Rightarrow \boxed{\omega^2 = \omega_c^2 + k^2 c_s^2}$$

acoustic modes, p-modes

2. $\omega^2 << S_l^2$

$$k_r^2 = \frac{\omega^2(\omega^2-\omega_c^2)c^2 + S_l^2(N^2-\omega^2)}{c^2\omega^2} =$$

$$= \frac{S_l^2(N^2-\omega^2)}{c^2\omega^2} = k_h^2\left(\frac{N^2}{\omega^2}-1\right) = k_h^2\frac{N^2}{\omega^2} - k_h^2$$

$$k_r^2 + k_h^2 = k_h^2 \frac{N^2}{\omega^2} = k^2$$

$$\boxed{\omega^2 = N^2 \frac{k_h^2}{k^2} = N^2 \cos^2\theta}$$

internal gravity modes / g-modes

propagate mostly horizontally

$\boxed{\text{p-modes}}$ $k_r^2 > 0$, for $k_r^2 = 0$ they reflect

for $\omega \gg \omega_c$, $\omega^2 \gg N^2$

equation for the lower turning point $k_r^2 = 0$

$$\omega^2 = c^2 \frac{S_\ell^2}{c^2\omega^2}\,\omega^2 = c^2\left[\sqrt{\frac{\ell(\ell+1)}{r}}\right]^2$$

$$\omega = \frac{c}{r}\sqrt{\ell(\ell+1)} = \frac{cL}{r}$$

$$\boxed{\frac{\omega}{L} = \frac{c(r_1)}{r_1}} \qquad r_1 \equiv r_t$$

upper turning point: $\omega_c(L) \sim \omega$

ω_c steep at the surface

$\rightarrow r_2 = R_\odot$

resonance:

$$\int_{r_1}^{r_2} k_r\,dr = \int_{r_1}^{r_2} \omega\sqrt{\frac{1}{c^2} - \frac{\ell(\ell+1)}{r^2\omega^2}}\,dr = \pi(n+\alpha)$$

$$\rightarrow \frac{\pi(n+\alpha)}{\omega} = \int_{r_t}^{R_\odot}\left[\frac{1}{c^2} - \frac{\ell(\ell+1)}{r^2\omega^2}\right]^{-1/2} dr$$

f-mode – in near-surface layers, $\delta p \sim 0$

$$p' = \delta p + g\rho\,\xi_r = \delta p + \frac{\partial p}{\partial r}\xi_r \quad \text{for Lagrange perturbation}$$

① $$\frac{d\xi_r}{dr} + \left[1 - \frac{\ell(\ell+1)c^2}{r^2\omega^2}\right]\frac{p'}{\rho c^2} = 0 =$$

$$= \frac{d\xi_r}{dr} + \left[1 - \frac{\ell(\ell+1)c^2}{r^2\omega^2}\right]\frac{\delta p}{\rho c^2} + \left[1 - \frac{\ell(\ell+1)c^2}{r^2\omega^2}\right]\frac{g\rho\,\xi_r}{\rho c^2} = 0$$

$$\frac{d\xi_r}{dr} - \xi_r\left[-\frac{g}{c^2} + \frac{\ell(\ell+1)c^2}{r^2\omega^2}\frac{g}{c^2}\right] + \left[1 - \frac{\ell(\ell+1)c^2}{r^2\omega^2}\right]\frac{\delta p}{\rho c^2} = 0$$

$$\frac{d\xi_r}{dr} - \xi_r\left[-\frac{g}{c^2} + \frac{\ell(\ell+1)}{r^2\omega^2}g\right] + \left[1 - \frac{\ell(\ell+1)}{r^2\omega^2}c^2\right]\frac{\delta p}{\rho c^2} = 0$$

② $$\frac{dp'}{dr} + \frac{g}{c^2}p' + (N^2 - \omega^2)\rho\,\xi_r = 0$$

$$\frac{\partial\,\delta p}{\partial r} + \frac{\partial(g\rho\xi_r)}{\partial r} + (N^2-\omega^2)\rho\xi_r + \frac{g}{c^2}\delta p + \frac{g}{c^2}g\rho\,\xi_r = 0$$

= XII =

$$\frac{\partial \delta_p}{\partial r} + g\rho \left(\frac{d\xi_r}{dr}\right) + \xi_r \frac{\partial g\rho}{\partial r} + (N^2 - \omega^2)\rho\, \xi_r + \frac{g}{c^2}\delta p + \frac{\rho}{c^2} g \rho\, \xi_r = 0$$

from ①

$$\frac{\partial \delta p}{\partial r} + g\rho\left[\xi_r\left(-\frac{g}{c^2} + \frac{l(l+1)}{r^2\omega^2} g\right) - \left(1 - \frac{l(l+1)}{r^2\omega^2} g\right) - \left(1 - \frac{l(l+1)}{r^2\omega^2} c^2\right)\frac{\delta p}{\rho c^2}\right] +$$

$$+ \xi_r \frac{\partial g\rho}{\partial r} + (N^2 - \omega^2)\rho\, \xi_r + \frac{g}{c^2}\delta p + \frac{g^2}{c^2}\rho\, \xi_r = 0$$

$$\frac{\partial \delta p}{\partial r} - g\rho\left(1 - \frac{l(l+1)c^2}{r^2\omega^2}\right)\frac{\delta p}{\rho c^2} + g\rho\, \xi_r\left(-g/c^2 + \frac{l(l+1)}{r^2\omega^2} g\right) +$$

$$+ \xi_r \frac{\partial g\rho}{\partial r} + (N^2 - \omega^2)\xi_r \rho + \frac{g}{c^2}\delta p + \frac{g^2}{c^2}\rho\, \xi_r = 0$$

$$\frac{\partial \delta p}{\partial r} + \delta p\left(\cancel{-g/c^2} + \frac{l(l+1)g}{r^2\omega^2} \cancel{+ g/c^2}\right) +$$

$$+ \xi_r\left[\cancel{-\frac{g^2\rho}{c^2}} + \frac{g^2\rho\, l(l+1)}{r^2\omega^2} + \frac{g\,\partial \rho}{\partial r} + (N^2 - \omega^2)\rho + \cancel{g^2/c^2 \rho}\right] = 0$$

$$\frac{\partial \delta p}{\partial r} + \frac{g\, l(l+1)}{\omega^2 r^2}\delta p + \xi_r \frac{g\rho}{r}\left[\frac{g\, l(l+1)}{\omega^2 r} + \underbrace{\frac{r}{\rho}\frac{\partial \rho}{\partial r} + \frac{N^2 r}{g}}_{(*)} - \frac{\omega^2 r}{g}\right] = 0$$

$$(*)\quad \frac{r}{g}\frac{\partial \rho}{\partial r} + \frac{N^2 r}{g} = \frac{r}{g}\left(\frac{g}{\rho}\frac{\partial \rho}{\partial r} + N^2\right) =$$

$$N^2 = g\left[\frac{1}{\lambda p}\frac{\partial p}{\partial r} - \frac{1}{\rho}\frac{\partial \rho}{\partial r}\right] = g\left[\frac{\rho}{\lambda p}\frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{1}{\rho}\frac{\partial \rho}{\partial r}\right] =$$

$$= g\left[\frac{1}{\rho c^2}\underbrace{\frac{\partial p}{\partial r}}_{\rho g} - \frac{1}{\rho}\frac{\partial \rho}{\partial r}\right] = g\left[g/c - \frac{1}{\rho}\frac{\partial \rho}{\partial r}\right]$$

$$= \frac{r}{g}\left(\cancel{\frac{g}{\rho}\frac{\partial \rho}{\partial r}} + g^2/c^2 - \cancel{\frac{g}{\rho}\frac{\partial \rho}{\partial r}}\right) = \frac{rg}{c^2} \leftarrow \text{constant}$$

for l large & ω large:

$$\frac{\partial \delta_p}{\partial r} + \frac{g\, l(l+1)}{\omega^2 r^2}\delta_p + \frac{g\rho}{r}\xi_r\Bigg(\underbrace{\frac{g(l+1)l}{\omega^2 r} - \frac{\omega^2 r}{g}}_{-f}\Bigg) = 0$$

$$\frac{\partial \delta p}{\partial r} + \frac{g\, l(l+1)}{\omega^2 r^2}\delta p - \frac{g\rho f}{r}\xi_r = 0$$

for $\nabla P = 0$ & $f = 0$ the equation can hold true

$$\Rightarrow f = 0 = \frac{g\,l(l+1)}{\omega^2 r} - \frac{\omega^2 r}{g} \quad / \cdot \omega^2 r g$$

$$g^2 l(l+1) - \omega^4 r^2 = 0$$

$$\boxed{\omega^2 = \frac{g}{r}\sqrt{l(l+1)} = k_h g} \quad \text{for } r = R_0$$

from ① for $\nabla P = 0$ and using

$$\frac{d\xi_r}{dr} - \frac{\sqrt{l(l+1)}}{r}\xi_r = 0$$

$$\Rightarrow \xi_r \sim e^{k_h(r - R_0)}$$

↳ exponential decay with depth