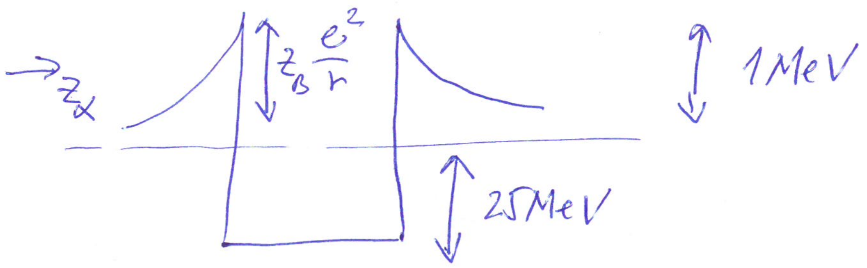
Gamov peak

→ slow reactions → yet ⊕ + ⊕ fuse



reaction rate per unit mass

$r_{\alpha B} \propto n_\alpha n_B \langle \sigma v \rangle$,

where $\langle \sigma v \rangle = \int \sigma v \frac{dn}{n}$

$\frac{dn}{n}$ → relative particle number in interval $[E, E+dE]$

the flux of particles α-type ($n_\alpha v$) onto particles B-type with effective cross-section σ

$\frac{dn}{n}$ – from the distribution function, assuming equilibrium ← Maxwell-Boltzmann

$$\frac{dn}{n} = \frac{2}{\sqrt{\pi}} \frac{1}{(kT)^{3/2}} e^{-\frac{E}{kT}} E^{1/2} dE$$

then: $r_{\alpha B} \propto \int \sigma v e^{-E/kT} E^{1/2} dE \propto \left| E = \frac{1}{2} m v^2 \right|$

$\propto \int \sigma E^{1/2} e^{-E/kT} E^{1/2} dE \propto \int \sigma E e^{-E/kT} dE$

effective cross-section → estimate from the de Broglie wavelength and correct for the tunnelling effect

de Broglie: $\lambda_p = \frac{h}{p} = \frac{h}{\sqrt{2mE}}$

$\sigma \propto \lambda_p^2$ times propability of the tunnelling, which is $\propto \exp\left(\frac{E_e}{E}\right)$, where

$$E_e = \frac{Z_\alpha Z_B e^2}{\lambda_p}$$

then $\sigma \propto \lambda_p^2 e^{-\frac{Z_\alpha Z_B e^2}{\lambda_p E}} = \frac{h^2}{2mE} \exp\left[-\frac{Z_\alpha Z_B e^2}{hE}\sqrt{2mE}\right] =$

$$= \frac{h^2}{2mE} \exp\left[-\frac{Z_\alpha Z_B e^2}{h}\frac{\sqrt{2m}}{\sqrt{E}}\right] \propto \frac{1}{E}\exp\left[-\frac{b}{E^{1/2}}\right]$$

$$b = \frac{Z_\alpha Z_B e^2 \sqrt{2m}}{h}$$

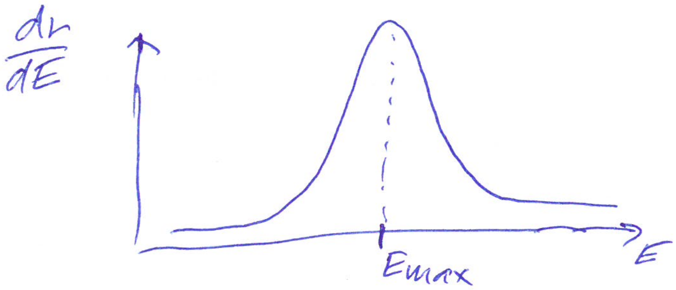
then the rate: $r \propto \int e^{\left(-E/kT - \frac{b}{E^{1/2}}\right)} dE$

$\frac{dr}{dE}$ – "efficiency" of the rates according to the energy

we search for the maximum

$$\frac{d}{dE}\left(\frac{dr}{dE}\right) = 0 = e^{\left(-\frac{E_{max}}{kT} - \frac{b}{E_{max}^{1/2}}\right)}\left[-\frac{1}{kT} + \frac{1}{2} b E_{max}^{-3/2}\right]$$

$$\Rightarrow \frac{2}{kTb} = E_{max}^{-3/2} \Rightarrow \underbrace{E_{max} \propto \left(\frac{bkT}{2}\right)^{2/3}}_{\text{gamov peak}}$$



estimate: Gamov peak narrow $\Rightarrow$ other energies almost do not contribute
$\rightarrow$ integrand can be replaced by narrow gaussian

then: $r \propto \frac{1}{T^{2/3}} e^{-\frac{a}{T^{1/3}}}$, $a = \frac{3(b/2)^{2/3}}{k^{1/3}}$

from M-B distribution
+ factor from the integration

Estimates from the internal structure

→ central temperature:

hydrostatic equilibrium + equation of state

$$\langle \rho \rangle \propto \frac{M_\odot}{R_\odot^3}$$

$$\frac{dp}{dr} \sim \frac{0 - P_c}{R_\odot - 0} \sim - \frac{G M_\odot \langle \rho \rangle}{(R_\odot - 0)^2} \sim - \frac{G M_\odot^2}{R_\odot^5}$$

$$P_c \sim \frac{\mathcal{R} \langle \rho \rangle T_c}{\mu} \sim \frac{\mathcal{R} M_\odot \boxed{T_c}}{\mu R_\odot^3}$$

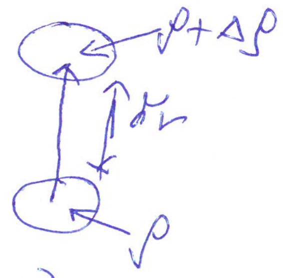
$$\Rightarrow \boxed{T_c} = \frac{P_c \mu R_\odot^3}{\mathcal{R} M_\odot} \sim \frac{G M_\odot^2}{R_\odot^4} \frac{\mu R_\odot^3}{\mathcal{R} M_\odot} \sim \frac{G M_\odot \mu}{\mathcal{R} R_\odot}$$

for $\mu = 0{,}6$ $\quad T_c \sim 1{,}4 \times 10^7$ K

Convection

surroundings bubble

$\rho' + \Delta\rho'$



ρ'

Mixing - length theory

- energy transport by convection

equation of motion for the bubble:

$\rho' > \rho \rightarrow$ rise...

$$\rho \frac{d^2 \delta r}{dt^2} = -g \Delta\rho = -g \left[\left(\frac{d\rho}{dr}\right)_{ad} - \left(\frac{d\rho}{dr}\right)_{rad} \right] \delta r$$

$$\rho = \frac{\mu P}{RT} \, ; \quad \frac{d\mu}{dr} \overset{!}{=} 0 \Rightarrow \frac{d\rho}{dr} = \frac{\mu}{RT} \frac{dP}{dr} - \frac{\mu P}{R} \frac{1}{T^2} \frac{dT}{dr}$$

$$\left(\frac{d\rho}{dr}\right)_{ad} - \left(\frac{d\rho}{dr}\right)_{rad} = \cancel{\frac{\mu}{RT} \frac{dP}{dr}} - \left(\frac{\mu P}{RT^2} \frac{dT}{dr}\right)_{ad} -$$

$$- \left[\cancel{\frac{\mu}{RT} \frac{dP}{dr}} - \left(\frac{\mu P}{RT^2} \frac{dT}{dr}\right)_{rad} \right] =$$

$$= -\frac{\mu P}{RT^2} \left[\left(\frac{dT}{dr}\right)_{ad} - \left(\frac{dT}{dr}\right)_{rad} \right] =$$

$$= \frac{\mu P}{RT^2} \left[\left(\frac{dT}{d\ln P}\right)_{ad} \frac{d\ln P}{dr} - \left(\frac{dT}{d\ln P}\right)_{rad} \frac{d\ln P}{dr} \right] =$$

$$= -\frac{\mu P}{RT^2} \underbrace{\left(\frac{d\ln P}{dr}\right)}_{-\frac{1}{H_p}} T \left[\left(\frac{d\ln T}{d\ln P}\right)_{ad} - \left(\frac{d\ln T}{d\ln P}\right)_{rad} \right] =$$

$$= + \overset{\rho}{\left(\frac{\mu P}{RT}\right)} \frac{1}{H_p} \left[\left(\frac{d\ln T}{d\ln P}\right)_{ad} - \left(\frac{d\ln T}{d\ln P}\right)_{rad} \right]$$

$$\rho \frac{d^2 \delta r}{dt^2} = -g \rho \frac{1}{H_p} \left[\underbrace{\left(\frac{d\ln T}{d\ln P}\right)_{ad}}_{\nabla_{ad}} - \underbrace{\left(\frac{d\ln T}{d\ln P}\right)_{rad}}_{\nabla} \right] \delta r$$

$$\frac{d^2 \delta r}{dt^2} = -\frac{g}{H_p} \left(\nabla_{ad} - \nabla\right) \delta r = -N^2 \delta r$$

$$N^2 = \frac{g}{H_p} \left(\nabla_{ad} - \nabla\right)$$

$\hookrightarrow$ Brunt-Väisälä frequency for $N^2 < 0$ → growth ⇒ convective instability

$N > 0$ → oscillating solution → g-modes

$$\frac{d^2 \delta r}{dt^2} = -\frac{g}{H_p}(\nabla_{ad} - \nabla)\,\delta r \qquad \Big/ \cdot 2\frac{d\delta r}{dt}$$

$$\frac{d}{dt}\left(\frac{d\delta r}{dt}\right)^2 = +\frac{g}{H_p}(\nabla - \nabla_{ad})\frac{d\,\delta r^2}{dt} \qquad \Big/ \int dt$$

$$\Rightarrow \left(\frac{d\delta r}{dt}\right)^2 = \frac{g}{H_p}(\nabla - \nabla_{ad})\,\delta r^2$$

introduce mixing-length → element travels without being destroyed to the distance ℓ, then it merges with surrounding. We set ℓ so that $\overline{\delta r} = \ell/2$

→ mean convective velocity $\frac{d\overline{\delta r}}{dt} = \bar{v}$

then $\bar{v}^2 = \frac{g}{4H_p}(\nabla - \nabla_{ad})\,\ell^2$

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energy flux: $F_c = \rho v^2 \cdot v \approx \rho\left[\frac{g}{4H_p}(\nabla - \nabla_{ad})\,\ell^2\right]^{3/2}$

deep in convection zone:

$\nabla > \nabla_{ad}$, but not much $\frac{\nabla - \nabla_{ad}}{\nabla} \ll 1$

⇒ slow velocities, time goes slow, all in local equilibrium

⇒ zone of efficient convection

under the surface $\frac{d\rho}{dr}$ steep decrease

⇒ large velocities, $\nabla \gg \nabla_{ad}$, speeds approach the speed of sound

⇒ inefficient convection

⇒ superadiabatic zone

(there is more than just the adiabatic convection which transports the energy)

in the mixing-length theory

$$\boxed{\ell = \alpha H_p}\,, \qquad \alpha \sim 1$$

↳ free parameter of the model

„universal constant"

Convection with radiative losses

the bubble irradiates during its rise

then $\underbrace{\nabla'}_{\text{real } \nabla} \neq \nabla_{ad}$; $\bar{v}^2 = \frac{g}{4H_p}(\nabla - \nabla')\ell^2$

temperature change within the bubble:

$$\Delta T = \Big[\underbrace{\left(\frac{dT}{dr}\right)'}_{\substack{\uparrow \\ \text{gradient inside the bubble} \\ \text{(not adiabatic, because it irradiates)}}} - \underbrace{\frac{dT}{dr}}_{\hookrightarrow \text{ surroundings}} \Big] \delta r$$

with respect to the surroundings

we keep using $\alpha = \ell / H_p$

convective flux: $F_c = \underbrace{\Delta T \rho c_p}_{\substack{\text{energy content} \\ = \text{heat} \leftarrow \text{calorimetric equation}}} \underbrace{\bar{v}}_{\text{speed}}$

$$(**)\quad \Delta T = \left[\left(\frac{dT}{dr}\right)' - \frac{dT}{dr} \right] \delta r = (\nabla - \nabla') \frac{T \delta r}{H_p}$$

$$\nabla' = -H_p \frac{d \ln T}{dr}$$

$$\delta r = \ell/2 \,,\quad \ell = \alpha H_p \Rightarrow \delta r = \frac{\alpha H_p}{2}$$

$$\Rightarrow \Delta T = (\nabla - \nabla') \frac{T\alpha}{2}$$

$$\Rightarrow F_c = \alpha \rho c_p T \bar{v} \frac{(\nabla - \nabla')}{2}$$

radiative loss from the bubble → from structural equations
→ equation of heat ballance

$$(*)\quad F_R' = -\frac{16 \sigma T^3}{3 \kappa \rho} \frac{\Delta T}{d} = \frac{\delta \alpha \sigma T^4}{3 \kappa \rho d} (\nabla' - \nabla)$$

d ... distance at which $\Delta T \rightarrow 0$
→ it corresponds to the size of the cell

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total convective flux → adiabatic + correction

$$F_c = F_c^{ad} + F_c' = \alpha \rho c_p v T (\nabla - \nabla_{ad})/2 + \alpha \rho c_p v T (\nabla_{ad} - \nabla')/2$$

only splitting in two contributions

super-adiabacity obvious now

„convective deficite"

within the cell – equilibrium

$F_R' = F_C'$ [convective deficite is radiated out]

hence: $$\boxed{\frac{8\sigma T^4}{3\varkappa \rho d}(\nabla' - \nabla) = \rho c_p \bar{v} T (\nabla_{ad} - \nabla')/2}$$

$$\hookrightarrow \bar{v} = \left[\frac{g}{4H_p}(\nabla - \nabla')\ell^2\right]^{1/2}$$

total energy ballance

$$F_R + F_C = \frac{L_\odot}{4\pi r^2}$$

$$\boxed{\frac{16\sigma T^4}{3\varkappa \rho H_p}\nabla + \alpha \rho c_p T \ell \sqrt{\frac{g}{4H_p}}\left(\frac{\nabla - \nabla'}{2}\right)^{3/2} = \frac{L_\odot}{4\pi r^2}}$$

$$\frac{\Delta T}{d} = \frac{\nabla T d}{H_p} \cdot \frac{1}{d} = T\frac{\nabla}{H_p}$$ analogy of (*) using (**)

two equations for ∇ and ∇', ∇_{ad} known

$$\frac{dp}{p} = \gamma \frac{d\rho}{\rho} \; ; \; p = A\rho \underbrace{T \Rightarrow}_{\text{divide}} dp = AT d\rho + A\rho\, dT$$

$$\frac{dp}{p} = \frac{ATd\rho + A\rho dT}{A\rho T} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

$$\nabla_{ad} = \left(\frac{d\ln T}{d\ln p}\right)_{ad} = \frac{p}{T}\left(\frac{dT}{dp}\right)_{ad}$$

$$1 = \frac{p}{\rho}\frac{d\rho}{dp} + \frac{p}{T}\left(\frac{dT}{dp}\right)_{ad} \Rightarrow \frac{p}{T}\left(\frac{dT}{dp}\right)_{ad} = 1 - \overset{1/\gamma}{\left(\frac{p}{\rho}\frac{d\rho}{dp}\right)} =$$

$$= 1 - 1/\gamma = \frac{\gamma - 1}{\gamma} = \nabla_{ad}$$

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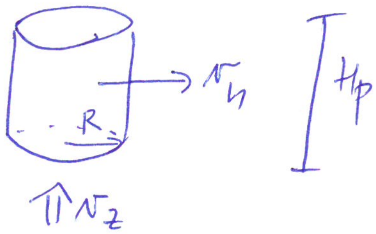
pozn: Ledoux ~~~~ parameter ~~term~~ of convective stability

$$A^* = \frac{1}{\gamma}\frac{d\ln p}{d\ln r} - \frac{d\ln \rho}{d\ln r}$$

$A^* < 0 \Leftarrow$ convective instability

The Sun: $\frac{dT}{dr}$ steep in outer layers $\to$ convection sets on to maintain the heat flux ($\sim T^3$)

Convective scales



continuity equation:

$$\pi R^2 v_z \rho \sim 2\pi R H_p \rho v_h$$

$$\Rightarrow R \sim 2 H_p \frac{v_h}{v_z}$$

estimates:

ballance of radiative losses and enthalpy flux

$H = U + pV$ ← enthalpy: energy content in the thermodynamical system

$$\sigma T_{eff}^4 \sim \rho v_z H$$

for hydrogen plasma: $H = 5/2\, kT + x\chi$

χ ... ionisation potential of hydrogen

x ... relative fraction of ionised hydrogen

$$\Rightarrow v_z = \frac{\sigma T_{eff}^4}{\rho H}$$

for $x \sim 0.1$ and solar values at the surface $v_z \sim 2$ km/s

horizontal speed – upper limit by sound speed $c_s \sim 7$ km/s in the photosphere

$$\Rightarrow 2R = \underbrace{4\,\mathrm{Mm}} \quad \text{for } H_p = 300\,\mathrm{km}$$

upper limit for the horizontal extend of convective scales on thermal-dissipation scale

→ depth dependence

$$R \sim 2H_p \frac{v_h}{v_z} = 2H_p \frac{c_s}{v_z} = 2H_p \frac{c_s}{\frac{\sigma T_{eff}^4}{\rho(5/2\, kT + x\chi)}} =$$

$$= 2H_p \frac{c_s (5/2\, kT + x\chi)\rho}{\sigma T_{eff}^4}$$

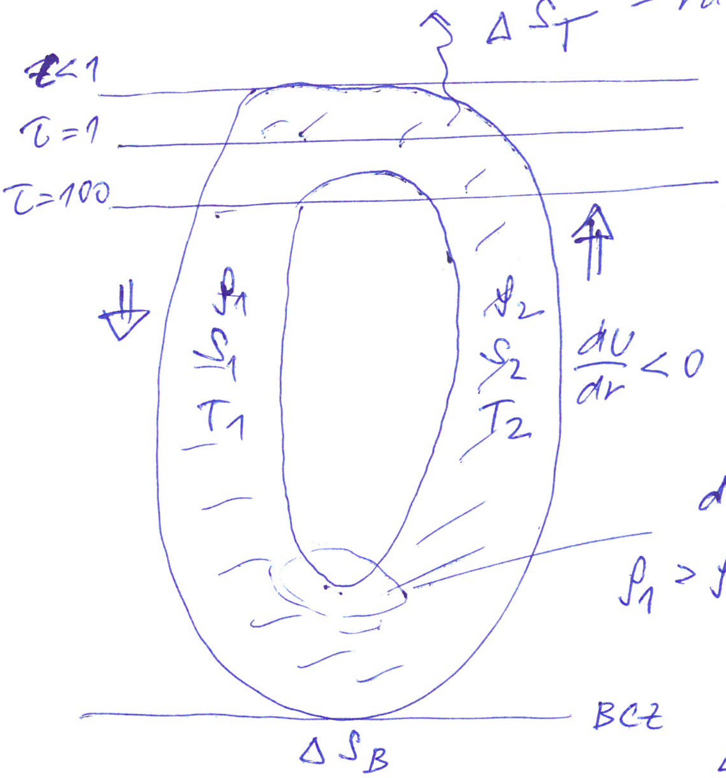
$$\cancel{a}\,\frac{dR}{dr} = A\,\frac{dH_p}{dr} + B. + \frac{dc_s}{dr} + C\,\frac{dT}{dr} + D.\,\frac{d\rho}{dr}$$

in the Sun: $\frac{\partial H_p}{\partial r} < 0$, $\frac{\partial c_s}{\partial r} < 0$, $\frac{\partial T}{\partial r} < 0$, $\frac{\partial \rho}{\partial r} < 0$

$$A, B, C, D > 0$$

$\Rightarrow \underline{\frac{\partial R}{\partial r} < 0}$ – with height the typical horizontal scale of convective cells decreases

$\boxed{\text{Convection}}$



for $T_2 < T_{ion}$ → recombination, fast cooling of the material by photons

drag & mixing → S_2 rises

$\rho_1 > \rho_2$, $S_1 < S_2$, $T_1 < T_2$

$\Delta S_T \gg \Delta S_B$

motion directed from the surface, where entropy fluctuations are largest (larger than at BCZ) most of the work is done by sinking plasma

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until 20 Mm depth → temperature rise from 4300 K → 143 000 K
density by 5.5 orders of mag.
pressure by 7 orders

20 Mm → BCZ – ↗ similarly fast ~~decrease~~ increase in temperature and pressure

$\Rightarrow$ all important at the surface!