CONSTRUCTION OF NEWCOMB OPERATORS ON A DIGITAL COMPUTER[1]

by

I. G. Izsak,[2] J. M. Gerard,[3] R. Efimba,[3] and M. P. Barnett[3]

Introduction

The first and most obvious application of modern electronic computers to problems of celestial mechanics was straightforward numerical integration of the equations of motion. Orbit determination, differential orbit improvement, computation of special perturbations or of general perturbations from given literal expressions are today carried out routinely at many computing centers, especially in connection with space research. A further, highly desirable, step toward automation of practical celestial mechanics would consist in developing computer programs that could produce the literal expressions for general perturbations.

It is well known that the algebra of any serious perturbation theory is formidable. Our present knowledge of motions in the solar system is still based largely on the planetary and lunar theories that were products of the exceptional capabilities of Leverrier, Delaunay, Hansen, Newcomb, Hill, and Brown. In the not very far future such analytical theories will certainly have to be improved. The only promising way of doing this is by progress in technology.

The analytical development of the disturbing (or perturbative) function is clearly the beginning of the construction of a general perturbation theory. Indeed, the derivation of the first-order perturbations either in orbital elements or in coordinates amounts to little more than

---

[1] This work was supported in part by grant number NsG 87/60 from the National Aeronautics and Space Administration.

[2] Research and Analysis Division, Smithsonian Astrophysical Observatory, Cambridge, Massachusetts.

[3] Cooperative Computing Laboratory, Massachusetts Institute of Technology, Cambridge, Massachusetts.

that. Therefore in a joint project between the Cooperative Computing Laboratory of the Massachusetts Institute of Technology and the Smithsonian Astrophysical Observatory we have recently investigated the possibility of using a digital computer for the complex algebraic manipulations that lead to the analytical development of the planetary disturbing function. Discussions soon showed the need for a simple formulation of the problem that could be coded in programming languages now available. Accordingly, in part 1 of this report, we have written a brief exposition of the development of the planetary disturbing function that was adopted, based on the use of Newcomb operators. Part 2 deals with a generalization of Newcomb operators that are introduced here in connection with the problem of near commensurabilities in the restricted problem of three bodies. Part 3 describes the program that was used on the IBM 709 of the Cooperative Computing Laboratory to produce Newcomb operators relating to the mean anomaly of the inner planet up to the 12th degree in the orbital eccentricity. Part 4 presents the results in polynomial form obtained by means of a photocomposition technique. The material for the first two parts of this report was prepared at the SAO; for the third and fourth, at MIT.

1. The analytical development of the planetary disturbing function

The differential equations for the perturbations in the orbital elements or in the coordinates of a planet contain the partial derivatives of the disturbing function

$$R = Gm' \{(r^2 + r'^2 - 2rr' \cos \Phi)^{-\frac{1}{2}} - rr'^{-2} \cos \Phi\} ,$$

where G is the universal constant of gravitation; m' is the mass of the perturbing planet and r' its radius vector; r is the radius vector of the perturbed planet; and Φ stands for the angular distance between the planets as seen from the Sun. The first term in this expression is called the principal, the second the indirect (or complementary) part of the disturbing function. Because the latter is not symmetrical in r and r' , one distinguishes two cases, according as the outer or inner planet is the perturbing body. Quantities pertaining to the inner and outer planet will be indicated by the subscripts 1 and 2. Then the disturbing function for the perturbations of an inner planet by an outer one appears in the form

$$R = Gm_2 \{(r_1^2 + r_2^2 - 2r_1 r_2 \cos \Phi)^{-\frac{1}{2}} - r_1 r_2^{-2} \cos \Phi\} ,$$

and the disturbing function for the perturbations of an outer planet by an inner one is

$$R = Gm_1 \{(r_1^2 + r_2^2 - 2r_1 r_2 \cos \Phi)^{-\frac{1}{2}} - r_2 r_1^{-2} \cos \Phi \} .$$

The integration of the equations of motion by some method of successive approximations is made possible by the development of the disturbing function into a multiple Fourier series, the arguments of which are linear combinations of the angular orbital elements. Methods for the approximate integration will not be discussed here, but it is expedient to note that the question of the most appropriate perturbation method in a particular case is far from being closed. It happens too often that the most elegant methods are not used because of practical difficulties, and methods actually used in practice are not satisfactory from a theoretical point of view.

The literature of the analytical development of the planetary disturbing function is very extensive; suffice it to refer to the works of Leverrier (1855), Tisserand (1880), Boquet (1889), Newcomb (1895), Poincaré (1907), von Zeipel (1912, 1913), Plummer (1918), Andoyer (1923) and Sharaf (1955). We carry out the development in question conveniently in two steps. First the orbits of the planets are considered to be circular. In the second step we generate from the resulting development the complicated terms that account for the eccentricities of the orbits by means of the Newcomb operators.

For circular orbits the disturbing function is

$$R_0 = Gm_2 a_2^{-1} \{(1 + \alpha^2 - 2\alpha \cos \Psi)^{-\frac{1}{2}} - \alpha \cos \Psi \} , \quad \text{if } a < a' \tag{1}$$

and

$$R_0 = Gm_1 a_2^{-1} \{(1 + \alpha^2 - 2\alpha \cos \Psi)^{-\frac{1}{2}} - \alpha^{-2} \cos \Psi \} , \quad \text{if } a > a' \tag{2}$$

with $\alpha = a_1 a_2^{-1} < 1$ and $\cos \Psi = \cos \lambda_1 \cos \lambda_2 + \sin \lambda_1 \sin \lambda_2 \cos J$, where $\lambda_1 = \omega_1 + M_1$ and $\lambda_2 = \omega_2 + M_2$ are the mean longitudes of the planets reckoned from the ascending node of the inner orbit on the outer one, and J is the mutual inclination of the orbital planes. Using the notations

$$\psi = \lambda_1 - \lambda_2, \quad \varphi = \lambda_1 + \lambda_2, \quad \mu = \cos^2 \frac{J}{2}, \quad \nu = \sin^2 \frac{J}{2}$$

we can write

$$\cos \Psi = \mu \cos \psi + \nu \cos \varphi \ . \tag{3}$$

In the domain of interest R_0 is a regular analytic function of the variables a_1, a_2, J, ψ, φ, periodic and even in the arguments ψ, φ, and therefore representable by a double Fourier series of the form

$$R_o = B_{oo} + 2\sum_1^\infty B_{ko} \cos k\psi + 2\sum_1^\infty B_{o\ell} \cos \ell\varphi + 4\sum_1^\infty \sum_1^\infty B_{k\ell} \cos k\psi \cos \ell\varphi. \tag{4}$$

It is advantageous to introduce the unimodular complex variables $\xi = \exp i\psi$, $\eta = \exp i\varphi$, whereupon R_0 becomes the double Laurent-series

$$R_0 = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} B_{k\ell}\, \xi^k \eta^\ell , \quad \text{with } B_{k\ell} = B_{|k||\ell|} \tag{5}$$

The coefficients in the Fourier expansion of the function

$$(1 + \alpha^2 - 2\alpha \cos \Psi)^{-\frac{1}{2}} = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} b_{k\ell}(\alpha, J)\, \xi^k \eta^\ell, \quad \text{with } b_{k\ell} = b_{|k||\ell|}, \tag{6}$$

bear the name of Jacobi. Then comparison of equations (1), (2), (5), (6) and (3) gives the following relations between the coefficients $B_{k\ell}$ and $b_{k\ell}$:

$$\begin{aligned}
\text{if } a < a', \quad & B_{1,0} = B_{-1,0} = Gm_2 a_2^{-1} \left(b_{1,0} - \frac{\mu}{2}\alpha\right) \\
& B_{0,1} = B_{0,-1} = Gm_2 a_2^{-1} \left(b_{0,1} - \frac{\nu}{2}\alpha\right) \qquad (7) \\
\text{and} \quad & B_{k,\ell} = Gm_2 a_2^{-1}\, b_{k,\ell} \text{ for all other indices ;} \\
\text{if } a > a', \quad & B_{1,0} = B_{-1,0} = Gm_1 a_2^{-1} \left(b_{1,0} - \frac{\mu}{2}\alpha^{-2}\right) \\
& B_{0,1} = B_{0,-1} = Gm_1 a_2^{-1} \left(b_{0,1} - \frac{\nu}{2}\alpha^{-2}\right) \qquad (8) \\
\text{and} \quad & B_{k,\ell} = Gm_1 a_2^{-1}\, b_{k,\ell} \text{ for all other indices .}
\end{aligned}$$

In the case of a small enough inclination J, that is for $\nu < (1-\alpha)^2/4\alpha$, the Jacobi coefficients can be developed into a power series of the quantity ν. To this end put $\cos \Psi = \cos \psi + \nu(\cos \varphi - \cos \psi)$ and consider the binomial expansion

$$(1+\alpha^2-2\alpha\cos\psi)^{-\frac{1}{2}} = \{(1+\alpha^2-2\alpha\cos\psi) - 2\nu\alpha(\cos\varphi-\cos\psi)\}^{-\frac{1}{2}}$$

$$= \sum_{0}^{\infty}\begin{bmatrix}1/2\\ n\end{bmatrix}\nu^n(2\cos\varphi-2\cos\psi)^n\alpha^n(1+\alpha^2-2\alpha\cos\psi)^{-\frac{1}{2}-n}, \tag{9}$$

where

$$\begin{bmatrix}1/2\\ n\end{bmatrix} = (-1)^n\begin{pmatrix}-1/2\\ n\end{pmatrix}.$$

The coefficients in the Fourier expansion of the function

$$\alpha^n(1+\alpha^2-2\alpha\cos\psi)^{-\frac{1}{2}-n} = \sum_{-\infty}^{\infty} b_h^{\frac{1}{2}+n}(\alpha)\xi^h = b_0^{\frac{1}{2}+n} + 2\sum_{1}^{\infty} b_h^{\frac{1}{2}+n}\cos(h\psi) \tag{10}$$

$$\text{with } b_h^{\frac{1}{2}+n} = b_{|h|}^{\frac{1}{2}+n}$$

are called here the Laplace coefficients; note that on the left side of this equation the symbol $\frac{1}{2}+n$ means an exponent, but on the right just an upper index. As to the other trigonometric factor in equation (9), we have

$$(2\cos\varphi-2\cos\psi)^n = (\eta+\eta^{-1}-\xi-\xi^{-1})^n = (1-\xi\eta)^n(\eta^{-1}-\xi^{-1})^n$$

$$= \sum_{i=0}^{n}\sum_{j=0}^{n}(-1)^{n+i-j}\binom{n}{i}\binom{n}{j}\xi^{i+j-n}\eta^{i-j}$$

$$= \sum_{\ell=-n}^{n}\left\{\sum_{j=0}^{n-|\ell|}\binom{n}{|\ell|+j}\binom{n}{j}\xi^{|\ell|-n+2j}\right\}(-1)^{n+\ell}\eta^{\ell}. \tag{11}$$

Now the Jacobi coefficient $b_{k\ell}$ belongs to the term $\xi^k\eta^\ell$ in the double Laurent series obtained upon substitution of the expansions (10) and (11) into expression (9). Because $b_{k\ell} = b_{|k||\ell|}$ per definition, it is sufficient to deal with the case $k, \ell \geq 0$. Then the smallest value of the index n for which a term will contain a particular factor η^ℓ is $n = \ell$. Therefore putting $n = \ell + m$ one gets

$$b_{k\ell} = \sum_{m=o}^{\infty} (-1)^m \nu^{\ell+m} b_{k\ell m}(\alpha) \ ,$$

where (12)

$$b_{k\ell m}(\alpha) = \begin{bmatrix} 1/2 \\ \ell+m \end{bmatrix} \sum_{j=o}^{m} \binom{\ell+m}{j} \binom{\ell+m}{m-j} b_{k+m-2j}^{\frac{1}{2}+\ell+m} \ .$$

The foregoing development of the Jacobi coefficients converges slowly, and becomes even meaningless for the perturbations of several minor planets by Jupiter. Another development, conceived by Leverrier, and elaborated by Tisserand, is valid for all values of the inclination J. It requires the computation of Laplace coefficients pertaining to the upper index $\frac{1}{2}$, and the evaluation of certain polynomials that are functions of the quantities μ and ν. Consider the expansion

$$(1 + \alpha^2 - 2\alpha \cos \Psi)^{-\frac{1}{2}} = b_o^{\frac{1}{2}} + 2 \sum_{1}^{\infty} b_h^{\frac{1}{2}} \cos(h\Psi). \qquad (13)$$

Recalling equation (3), the trigonometric function $\cos(h\Psi)$ can be represented by a finite sum of the form

$$\cos(h\Psi) = Q_{oo}^{h} + 2\sum_{1}^{h} Q_{ko}^{h} \cos k\psi + 2\sum_{1}^{h} Q_{o1}^{h} \cos \ell\varphi + 4\sum_{1}^{h}\sum_{1}^{h} Q_{k\ell}^{h} \cos k\psi \cos \ell\varphi .$$

Here the coefficients $Q_{k\ell}^{h}$ are polynomials in μ and ν, with the understanding that $Q_{k\ell}^{h} = 0$, if $h-(k+\ell)$ is a negative or odd integer. Thus substitution of this sum into expansion (13) yields the infinite series

$$b_{oo} = b_o^{\frac{1}{2}} + 2 \sum_{j=1}^{\infty} Q_{oo}^{2j} \, b_{2j}^{\frac{1}{2}} \qquad \text{for } k = \ell = 0,$$

and (14)

$$b_{k\ell} = 2 \sum_{j=o}^{\infty} Q_{k\ell}^{k+\ell+2j} \, b_{k+\ell+2j}^{\frac{1}{2}} \qquad \text{for all other indices.}$$

Remarkably simple explicit expressions of the polynomials $Q^h_{k\ell}(\mu, \nu)$ and their derivatives with respect to J were found by Tisserand. He has observed that the polynomials $R^h_{k\ell}(\mu, \nu)$ in the representation

$$\frac{\sin(h+1)\Psi}{\sin\Psi} = R^h_{oo} + 2\sum_1^h R^h_{ko}\cos k\psi + 2\sum_1^h R^h_{o\ell}\cos \ell\varphi + 4\sum_1^h\sum_1^h R^h_{k\ell}\cos k\psi\cos\ell\varphi$$

are of a more transparent structure than the $Q^h_{k\ell}$. The identity

$$2\cos(h\Psi) = \frac{\sin(h+1)\Psi}{\sin\Psi} - \frac{\sin(h-1)\Psi}{\sin\Psi}$$

shows that

$$2Q^h_{k\ell} = R^h_{k\ell} - R^{h-2}_{k\ell}, \tag{15}$$

so that the evaluation of the polynomials $R^h_{k\ell}$ is equivalent to that of the $Q^h_{k\ell}$; clearly,

$$R^{-h}_{k\ell} = -R^{h-2}_{k\ell}, \quad \text{while} \quad Q^{-h}_{k\ell} = Q^h_{k\ell}.$$

Moreover, the relation

$$4\,dQ^h_{k\ell}/dJ = h\sin J\,(R^h_{k,\ell+1} + R^h_{k,\ell-1} - R^h_{k+1,\ell} - R^h_{k-1,\ell}) \tag{16}$$

is valid. Regarding the polynomials $R^h_{k\ell}$, Tisserand has recognized that they are expressible by the square of a hypergeometric polynomial

$$R^h_{k\ell} = \binom{j+k+\ell}{\ell}\binom{j+\ell}{\ell}\mu^k\nu^\ell F^2(-j, j+k+\ell+1, \ell+1\,|\,\nu), \tag{17}$$

where

$$j = \tfrac{1}{2}[h-(k+\ell)] \geq 0 .$$

The development (5) of the disturbing function for circular orbits with an arbitrary inclination being fully determined, we now proceed to generate the terms that correspond to the orbital eccentricities. Let us adopt the notations v_1, v_2 and $u_1 = \omega_1 + v_1$, $u_2 = \omega_2 + v_2$ for the true anomalies and true longitudes of the two planets, and introduce the complex variables

$$x_1 = \exp iv_1 \quad , \quad x_2 = \exp iv_2$$

$$\upsilon_1 = \exp iu_1 \quad , \quad \upsilon_2 = \exp iu_2 \quad .$$

In connection with the mean anomalies and mean longitudes we will use the symbols

$$z_1 = \exp iM_1 \quad , \quad z_2 = \exp iM_2$$

$$\zeta_1 = \exp i\lambda_1 \quad , \quad \zeta_2 = \exp i\lambda_2 \quad .$$

Since $\xi = \zeta_1 / \zeta_2$ and $\eta = \zeta_1 \zeta_2$, the development (5) can be written as

$$R_0 = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} B_{k\ell}(a_1, a_2, J)\zeta_1^{k+\ell}\zeta_2^{-k+\ell} \tag{18}$$

$$= B_{oo} + 2\sum_{1}^{\infty} B_{ko} \cos[k(\lambda_1 - \lambda_2)] + 2\sum_{1}^{\infty} B_{o\ell} \cos[\ell(\lambda_1 + \lambda_2)]$$

$$+ 2\sum_{1}^{\infty}\sum_{1}^{\infty} B_{k\ell}\{\cos[(k+\ell)\lambda_1 - (k-\ell)\lambda_2] + \cos[(k+\ell)\lambda_2 - (k-\ell)\lambda_1]\} .$$

For elliptical orbits, R is obviously the same function of the variables r_1, r_2, J, υ_1, υ_2 as R_0 was found to be of the variables a_1, a_2, J, ζ_1, ζ_2. In other terms,

if $R_0 = f[a_1, a_2, J \,|\, \zeta_1, \zeta_2]$,

then

$$R = f[r_1, r_2, J \,|\, \upsilon_1, \upsilon_2] = f[a_1(r_1/a_1), a_2(r_2/a_2), J \,|\, \zeta_1(x_1/z_1), \zeta_2(x_2/z_2)] . \tag{19}$$

The Fourier expansion of the radius vector and of the so-called equation of the center in terms of the mean anomaly are known from the theory of Keplerian motion; in the complex form their first terms are

$$\frac{r}{a} = \left(1 + \frac{e^2}{2}\right) - \left(\frac{e}{2} - \frac{3e^3}{16} + \ldots\right)(z + z^{-1}) - \left(\frac{e^2}{4} - \ldots\right)(z^2 + z^{-2}) - \left(\frac{3e^3}{16} - \ldots\right)(z^3 + z^{-3}) - \ldots$$

and

$$\frac{x}{z} = (1 - e^2 + \ldots) + \left(e - \frac{5e^3}{4} + \ldots\right)z - (e - 0 . e^3 + \ldots)z^{-1} + \left(\frac{9e^2}{8} - \ldots\right)z^2 - \left(\frac{e^2}{8} - \ldots\right)z^{-2} + \left(\frac{4e^3}{3} - \ldots\right)z^3 - \left(\frac{e^3}{12} - \ldots\right)z^{-3} + .$$

(An important feature of these expansions is that the coefficient of z^j contains $e^{|j|}$ as a factor.) In order to obtain the general development of the disturbing function one could use the above expansions directly for substitution in equation (19). This, however, would be an extremely tedious task. Perfecting the techniques of his predecessors, Newcomb has devised a symbolic method that achieves the required substitution in a most appropriate way. It depends on the use of certain differential operators, equivalent to the analytical expressions of the Hansen coefficients. The original presentation of this method by Newcomb has since been simplified on several occasions.

Before dealing with our actual problem, we consider first a simple example. Given the above Fourier expansion of r/a, let us seek that of an arbitrary analytic function $f[r] = f[a(r/a)]$. The function r/a being of the form

$$r/a = 1 + g(e, z), \quad \text{with} \quad g(e, z) = O(e) ,$$

we imagine $f[r]$ to be developed into the Taylor series

$$f[a(1 + g)] = \sum_0^\infty \frac{a^i}{i!} \frac{d^i f[a]}{da^i} \{g(e, z)\}^i .$$

This expression is conveniently thought of as the differential operator

$$\sum_0^\infty \{g(e, z)\}^i \frac{a^i}{i!} \frac{d^i}{da^i} , \tag{20}$$

acting upon the function $f[a]$. In terms of the differential operator $D = a\frac{d}{da}$ we have, by induction

$$a\frac{d}{da} = D, \quad a^2\frac{d^2}{da^2} = D(D-1), \; \ldots \;, \; a^i\frac{d^i}{da^i} = D(D-1)\ldots(D-i+1) \,,$$

so that the differential operator (20) appears in the form

$$\sum_0^{\infty} \{g(e, z)\}^i \binom{D}{i} \quad .$$

We rearrange this symbolic series according to the positive and negative powers of the argument z, for the expansion of the function f[r] is required in the form of a Laurent series. The coefficients of z^j will be power series in e with polynomial coefficients in D , which are obtained by putting D for n in the corresponding arrangement of the binomial series

$$\begin{aligned}(r/a)^n = \{1 + g(e,z)\}^n &= \sum_0^{\infty}\binom{n}{i}\{g(e,z)\}^i \\ &= \{1 + (n+n^2)\frac{e^2}{4} + \ldots\} \\ &\quad + \{(-n)\frac{e}{2} + (3n+n^2-n^3)\frac{e^3}{16} + \ldots\}(z + z^{-1}) \\ &\quad + \{(-3n+n^2)\frac{e^2}{8} + \ldots\}(z^2 + z^{-2}) \\ &\quad + \{(-17n+9n^2-n^3)\frac{e^3}{48} + \ldots\}(z^3 + z^{-3}) + \ldots \quad .\end{aligned} \tag{21}$$

For this reason, we write symbolically

$$f[r] = (r/a)^D f[a] \,, \quad \text{where} \quad D = a\frac{d}{da} \quad . \tag{22}$$

This formula already gives a better insight into the structure of the expansion of the function f[r]. Its practical advantage results from the circumstance that the coefficients of the various powers of the eccentricity in equation (21) can be constructed by using relatively simple recurrence relations.

The generalization of formula (22) to the case of several variables is immediate. For the development of the disturbing function we have

$$f[r_1, r_2, J \,|\, \upsilon_1, \upsilon_2] = (r_1/a_1)^{D_1}(r_2/a_2)^{D_2}(x_1/z_1)^{D_3}(x_2/z_2)^{D_4} f[a_1, a_2, J \,|\, \zeta_1, \zeta_2]$$

where $D_1 = a_1 \partial/\partial a_1$, $D_2 = a_2 \partial/\partial a_2$, $D_3 = \zeta_1 \partial/\partial \zeta_1$, $D_4 = \zeta_2 \partial/\partial \zeta_2$.

In analogy with equation (21), the expansion of the differential operator $(r_1/a_1)^{D_1}(x_1/z_1)^{D_3}$ will be of the form

$$\sum_{-\infty}^{\infty} \left\{ \sum_{m_1} \prod_{j_1}^{m_1} (D_1 | D_3)\, e_1^{m_1} \right\} z_1^{j_1} , \tag{23}$$

the second summation comprising all indices m_1 , for which $m_1 - |j_1| = 0, 2, 4, \ldots$. The coefficient

$$\prod_{j_1}^{m_1} (D_1 | D_3) ,$$

a polynomial in the symbols D_1, D_3 of degree m_1, specified by the indices j_1, m_1, is called a Newcomb operator relating to the inner planet. Similarly relating to the outer planet we write

$$(r_2/a_2)^{D_2}(x_2/z_2)^{D_4} = \sum_{-\infty}^{\infty} \left\{ \sum_{m_2} \prod_{j_2}^{m_2} (D_2 | D_4)\, e_2^{m_2} \right\} z_2^{j_2} , \tag{24}$$

where $m_2 - |j_2| = 0, 2, 4, \ldots$.

We consider next the effect of these Newcomb operators on the disturbing function (18) for circular orbits. The rule of differentiation

$$D_3^h \zeta_1^{k+\ell} = (k+\ell)^h \zeta_1^{k+\ell} \qquad (h = 0, 1, 2, \ldots)$$

shows that the symbol D_3 can be replaced by $k+\ell$, that is in so far as a particular term of R_o is concerned,

$$\prod_{j_1}^{m_1} (D_1 | D_3) = \prod_{j_1}^{m_1} (D_1 | k+\ell) .$$

Similarly,

$$\prod_{j_2}^{m_2} (D_2 | D_4) = \prod_{j_2}^{m_2} (D_2 | -k+\ell) .$$

Furthermore, R_o is homogeneous and of degree -1 in the variables a_1, a_2, and so are the coefficients $B_{k\ell}(a_1, a_2, J)$. Hence by Euler's formula we have

$$D_1 B_{k\ell} + D_2 B_{k\ell} = -B_{k\ell}, \quad \text{that is} \quad D_2 = -D_1 - 1 \ .$$

Let us denote the polynomial that becomes of

$$\prod_{j_2}^{m_2}(D_2 \mid -k+\ell) \quad \text{upon this substitution}$$

by

$$\,^{m_2}_{j_2}\prod(D_1 \mid -k+\ell) = \prod_{j_2}^{m_2}(-D_1 - 1 \mid -k+\ell) \ , \tag{25}$$

and introduce the combined Newcomb operators

$$\prod_{j_1\ j_2}^{m_1\ m_2}(D_1 \mid k,\ell) = \prod_{j_1}^{m_1}(D_1 \mid k+\ell) \cdot \,^{m_2}_{j_2}\prod(D_1 \mid -k+\ell) \tag{26}$$

relating to both planets. For the result of applying these operators to the functions $B_{k\ell}$ we write

$$C^{m_1\ m_2}_{j_1\ j_2 k\ell} = \prod_{j_1\ j_2}^{m_1\ m_2}(D_1 \mid k,\ell) B_{k\ell} \ . \tag{27}$$

In order to utilize the expressions (7), or (8), we note that

$$D_1^h B_{k\ell} = a_2^{-1} D^h (a_2 B_{k\ell}), \qquad (h = 0, 1, 2, \ldots)$$

where $\quad D = \alpha\,\partial/\partial\alpha \ , \quad$ and $\quad \alpha = a_1 a_2^{-1} \ .$

Thus

if $a < a'$,

$$D_1^h B_{10} = Gm_2 a_2^{-1}(D^h b_{10} - \frac{\mu}{2}\alpha) \tag{28}$$

$$D_1^h B_{01} = Gm_2 a_2^{-1}(D^h b_{01} - \frac{\nu}{2}\alpha)$$

and $\quad D_1^h B_{k\ell} = Gm_2 a_2^{-1} D^h b_{k\ell} \quad$ for all other indices;

if $a > a'$,

$$D_1^h B_{10} = Gm_1 a_2^{-1}(D^h b_{10} + \mu(-2)^{h-1}\alpha^{-2}) \tag{29}$$

$$D_1^h B_{01} = Gm_1 a_2^{-1}(D^h b_{01} + \nu(-2)^{h-1}\alpha^{-2})$$

and $$D_1^h B_{k\ell} = Gm_1 a_2^{-1} D^h b_{k\ell}$$ for all other indices.

The derivatives on the right sides of these formulas are linear combinations of the corresponding derivatives of Laplace coefficients,* as shown both by equations (12) and (14).

Finally, if for the sake of brevity we put

$$C_{j_1 j_2 k \ell} = C_{-j_1, -j_2, -k, -\ell} = \sum_{m_1} \sum_{m_2} C_{j_1 j_2 k \ell}^{m_1 m_2} e_1^{m_1} e_2^{m_2} , \tag{30}$$

then the general development of the planetary disturbing function is the quadruple Fourier series

$$R = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} C_{j_1 j_2 k \ell} \cos [j_1 M_1 + j_2 M_2 + (k+\ell) \lambda_1 - (k-\ell) \lambda_2] . \tag{31}$$

Here the coefficients contain

$$e_1^{|j_1|} e_2^{|j_2|} \sin^2 |\ell| \frac{J}{2}$$

as a factor, and the arguments can also be written in the form

$$(j_1 + k + \ell) \lambda_1 + (j_2 - k + \ell) \lambda_2 - j_1 \omega_1 - j_2 \omega_2 .$$

Our problem has therefore been completely reduced to constructing the combined operators

$$\prod_{j_1 \ j_2}^{m_1 \ m_2} (D_1 | k, \ell)$$

for the relevant values of their arguments k and ℓ. Newcomb himself deemed it advisable to express the symbols (26) for fixed values of $\ell = 0, 1, 2, \ldots$ as polynomials in the arguments D_1 and k, and then substitute into these algebraic expressions the required values of k. Von Zeipel pointed out, however, that this is much more difficult than to construct only the constituent operators

---

* An IBM 7090 computer program has been written at SAO to evaluate the Laplace coefficients and their Newcomb derivatives (Izsak and Benima, 1963).

$$\prod_{j_1}^{m_1}(D_1 | k) \quad \text{and} \quad {}^{m_2}_{j_2}\prod (D_1 | k)$$

as polynomials in the arguments D_1 and k, substitute the required numerical values of $k+\ell$ and $-k+\ell$ for k in the first and second operator, and then multiply the polynomials in the single variable D_1 so obtained.

He also disclosed the equivalence of the Newcomb operators $\prod_{j_1}^{m_1}(D_1 | k)$ to the analytical expressions of the Hansen coefficients, and established convenient recurrence relations to construct them, which Andoyer subsequently somewhat modified.

It is sufficient to treat the operators $\prod_{j_1}^{m_1}(D_1 | k)$, because upon substitution of $-D_1 - 1$ for D_1 and rearrangement according to powers of D_1 they easily yield the operators

$${}^{m_2}_{j_2}\prod (D_1 | k) = \prod_{j_2}^{m_2}(-D_1 - 1 | k) \quad .$$

Let us define with Hansen the (real) coefficients $X_j^{n,k}(e)$ by the expansion

$$(r/a)^n x^k = \sum_{-\infty}^{\infty} X_j^{n,k} z^j, \qquad (n, k = 0, \pm 1, \pm 2, \ldots)$$

or what is the same thing, by

$$X^{n,k}(e, z) = (r/a)^n (x/z)^k = \sum_{-\infty}^{\infty} X_{j+k}^{n,k} z^j \quad . \tag{32}$$

Considering them functions not only of the eccentricity e, but also of the superscripts n and k, their expression in terms of the Newcomb operators is immediately seen to be

$$X_{j+k}^{n,k}(e) = \sum_m \prod_j^m (n | k) e^m \quad .$$

The conjugates of the unimodular complex variables x and z being equal to their reciprocals, we have the relation of symmetry

$$X_{j+k}^{n,k} = X_{-j-k}^{n,-k} \text{ , and consequently,}$$

$$\prod{}_j^m(n|k) = \prod{}_{-j}^m(n|-k) \quad . \tag{33}$$

The recurrence relations of von Zeipel follow from the partial differential equation

$$(1-e^2)e\frac{\partial}{\partial e}X^{n,k} + (1-e^2)^{3/2}\, z\frac{\partial}{\partial z}X^{n,k}$$
$$= \{k[1-(1-e^2)^{3/2}] + (k-n)\frac{e^2}{2} + (2k-n)ex + (k-n)\frac{e^2}{2}x^2\}X^{n,k}$$

satisfied by the functions (32), the derivation of which will not be given here. Eliminating the variable x by the substitution $x = z(x/z)$ and developing into powers of e, we can write this differential equation as

$$2\{e\frac{\partial}{\partial e} + z\frac{\partial}{\partial z}\}X^{n,k} = 2(2k-n)ezX^{n,k+1} + (k-n)e^2z^2X^{n,k+2}$$
$$+ e^2\{(4k-n) + 2e\frac{\partial}{\partial e} + 3z\frac{\partial}{\partial z}\}X^{n,k} - 2\sum_{\tau\geq 2}\binom{3/2}{\tau}(-e^2)^\tau\{k+z\frac{\partial}{\partial z}\}X^{n,k} \quad . \tag{34}$$

To simplify indexing in the work with the recurrence relations we introduce the alternate notation

$$X^{n,k}(e,z) = \sum_0^\infty\sum_0^\infty X_{\rho,\sigma}^{n,k} e^{\rho+\sigma} z^{\rho-\sigma} \quad , \tag{35}$$

so that $\quad X_{\rho,\sigma}^{n,k} = \prod{}_{\rho-\sigma}^{\rho+\sigma}(n|k)$ and $\prod{}_j^m(n|k) = X^{n,k}_{\frac{m+j}{2},\,\frac{m-j}{2}}$. (36)

Since the polynomials $X_{\rho,\sigma}^{n,k}$ possess the symmetry

$$X_{\rho,\sigma}^{n,k} = X_{\sigma,\rho}^{n,-k} \quad , \tag{37}$$

only those with $\rho \geq \sigma$ have to be determined. Now enter the series (35) into the differential equation (34). Comparing the coefficients of corresponding terms, we obtain the recurrence relation

$$4\rho X_{\rho,\sigma}^{n,k} = 2(2k-n)X_{\rho-1,\sigma}^{n,k+1} + (k-n)X_{\rho-2,\sigma}^{n,k+2}$$
$$+ (5\rho-\sigma-4+4k-n)X_{\rho-1,\sigma-1}^{n,k} - 2(\rho-\sigma+k)\sum_{\tau\geq 2}(-1)^\tau\binom{3/2}{\tau}X_{\rho-\tau,\sigma-\tau}^{n,k} \quad . \tag{38}$$

It is understood that in this equation the subscripts cannot take negative values and therefore the number of terms on its right side is never more than $2+\min(\rho,\sigma)$. Another recurrence relation results from (38) if we interchange the indices ρ and σ, reverse the sign of the argument k, and make use of the property (37), namely

$$4\sigma X^{n,k}_{\rho,\sigma} = -2(2k+n)X^{n,k-1}_{\rho,\sigma-1} - (k+n)X^{n,k-2}_{\rho,\sigma-2} - (\rho-5\sigma+4+4k+n)X^{n,k}_{\rho-1,\sigma-1} + 2(\rho-\sigma+k)\sum_{\tau\geq 2}(-1)^{\tau}\binom{3/2}{\tau}X^{n,k}_{\rho-\tau,\sigma-\tau} . \tag{39}$$

The construction of the polynomials $X^{n,k}_{\rho,\sigma}$ starts with setting $X^{n,k}_{0,0} = 1$, and continues in the order

$$X^{n,k}_{1,0} = k - \frac{n}{2} ,$$

then $X^{n,k}_{2,0}$, $X^{n,k}_{1,1}$,

then $X^{n,k}_{3,0}$, $X^{n,k}_{2,1}$,

then $X^{n,k}_{4,0}$, $X^{n,k}_{3,1}$, $X^{n,k}_{2,2}$ . . . ,

etc. Equation (38) is used to determine the polynomials in the first column, and equation (39) in all other cases.

The coefficients of these polynomials are rational numbers. With the polynomials

$$J^{n,k}_{\rho,\sigma} = 2^{\rho+\sigma}\rho!\sigma!X^{n,k}_{\rho,\sigma} , \tag{40}$$

however, the work is confined to the realm of integer arithmetic. Their recurrence relations follow at once from equations (38) and (39), and the definition (40):

$$J^{n,k}_{\rho,0} = (2k-n)J^{n,k+1}_{\rho-1,0} + (\rho-1)(k-n)J^{n,k+2}_{\rho-2,0} \tag{41}$$

$$J^{n,k}_{\rho,\sigma} = -(2k+n)J^{n,k-1}_{\rho,\sigma-1} - (\sigma-1)(k+n)J^{n,k-2}_{\rho,\sigma-2} \tag{42}$$

$$-\rho(\rho-5\sigma+4+4k+n)\,J^{n,k}_{\rho-1,\sigma-1} + \rho(\rho-\sigma+k)\sum_{\tau\geq 2} c_{\rho\sigma\tau}\,J^{n,k}_{\rho-\tau,\sigma-\tau}\,,$$

where $c_{\rho\sigma\tau} = (\rho-1)(\rho-2)\ldots(\rho-\tau+1)\;(\sigma-1)(\sigma-2)\ldots(\sigma-\tau+1)c_\tau$,

$$\text{with } c_\tau = (-1)^\tau \binom{3/2}{\tau} 2^{2\tau-1} = 3,\ 2,\ 3,\ 6,\ 14,\ 36,\ 99,\ \ldots, \tag{43}$$

the range of summation being $\tau = 2, 3, \ldots, \min(\rho,\sigma)$.

Newcomb (1895) computed the operators relating to the inner and outer planet up to the 8th degree in the eccentricities by a method of his own. He also gave the algebraic expressions for the combined operators relating to both planets up to the 7th degree in the eccentricities and 2nd degree in the inclination. In a recent work on the theory of Pluto, Sharaf (1955) found several of Newcomb's 8th-degree operators relating to the outer planet in error and corrected them.

The first few Newcomb operators pertaining to the mean anomaly of the inner planet are:

$$
\begin{aligned}
\prod\nolimits_0^0(D|k) &= J^{D,k} = 1 \\
2\prod\nolimits_1^1(D|k) &= J^{D,k}_{1,0} = 2k - D \\
8\prod\nolimits_2^2(D|k) &= J^{D,k}_{2,0} = (5k+4k^2) - (3+4k)D + D^2 \\
4\prod\nolimits_0^2(D|k) &= J^{D,k}_{1,1} = -4k^2 + D + D^2 \\
48\prod\nolimits_3^3(D|k) &= J^{D,k}_{3,0} = (26k+30k^2+8k^3) - (17+33k+12k^2)D + (9+6k)D^2 - D^3 \\
16\prod\nolimits_1^3(D|k) &= J^{D,k}_{2,1} = -(2k+10k^2+8k^3) + (3+5k+4k^2)D + (1+2k)D^2 - D^3 \\
384\prod\nolimits_4^4(D|k) &= J^{D,k}_{4,0} = (206k+283k^2+120k^3+16k^4) - (142+330k+192k^2+32k^3)D \\
&\qquad + (95+102k+24k^2)D^2 - (18+8k)D^3 + D^4 \\
96\prod\nolimits_2^4(D|k) &= J^{D,k}_{3,1} = -(22k+64k^2+60k^3+16k^4) + (22+47k+48k^2+16k^3)D \\
&\qquad - (1-3k)D^2 - (6+4k)D^3 + D^4 \\
64\prod\nolimits_0^4(D|k) &= J^{D,k}_{2,2} = -(9k^2-16k^4) + 2D - (1+8k^2)D^2 - 2D^3 + D^4 \ldots .
\end{aligned}
\tag{44}
$$

Here we took the liberty of writing k for Newcomb's $-i$.

2. Generalized Newcomb operators

The Newcomb operators, as usually understood, are connected with the development of the planetary disturbing function in terms of the Keplerian orbital elements. But nothing prevents us from constructing analogous differential operators when this development is required in terms of some canonical variables. Such is the case in many theoretical problems of celestial mechanics. As an example, we choose the question of small divisors in the planetary restricted problem of three bodies, the proper understanding of which is a first step toward successful orbital theories for minor planets having mean motions nearly commensurable with that of Jupiter.

Let the units of length, mass, and time be, respectively, the mean distance of Jupiter, the mass of the Sun and 689.8817 ephemeris days. In these units the gravitational constant G becomes equal to 1, and the mass and mean motion of Jupiter are

$$m_J = 1/1047.355 \quad \text{and} \quad n_J = (1+m_J)^{\frac{1}{2}} = 1.000477 \text{ radians.}$$

The minor planet of negligible mass moves in the orbital plane of Jupiter, while the orbit of the latter is supposed to be circular. In terms of the Delaunay variables

$$L = a^{\frac{1}{2}}\ ,\ G = [a(1-e^2)]^{\frac{1}{2}}\ ,\ M\ ,\ \omega \tag{45}$$

the motion of the minor planet is described by the canonical equations

$$\dot{L} = F_M = R_M,\ \dot{G} = F_\omega = R_\omega,\ \dot{M} = -F_L = L^{-3} - R_L,\ \dot{\omega} = -F_G = -R_G \tag{46}$$

associated with the Hamiltonian function

$$F = 1/2L^2 + R\ ,$$

and the development of the disturbing function:

$$R = m_J\{(1 + r^2 - 2r\cos\Phi)^{-\frac{1}{2}} - r\cos\Phi\} \tag{47}$$

is of the form

$$R = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} C_{jk}(L, G)\cos(jM+k\psi)\ , \tag{48}$$

where $\psi = \lambda - \lambda_J \ , \quad \lambda = \omega + M \ , \quad \lambda_J = n_J t \ .$

Needless to say, the quantity $L^{-3} = a^{-3/2} = n$ in the third of equations (46) is the (osculating) mean motion of the minor planet.

For $m_J = 0$, the Keplerian solutions to the differential system (46) are simply

$$L = L(0), \quad G = G(0), \quad M = M(0) + n(0)t, \quad \omega = \omega(0).$$

If $m_J \neq 0$, but a small parameter, various perturbation methods can be applied to obtain approximate solutions. Astronomical experience shows that planetary perturbations have two qualitatively different components. Observations made over a relatively short interval of time reveal small, multiply periodic oscillations of the orbital elements. These so-called short-periodic perturbations in turn are superposed on smooth and steady variations of the mean elements, reaching considerable amounts in the course of centuries. It is the latter, long-periodic and secular perturbations that cause serious mathematical difficulties in orbital theories valid for very extended times. The foregoing "definitions" are very difficult, if not impossible, to formulate in a precise way; there is an almost continuous transition between the perturbations of different kinds. In addition, the nature of the long-range perturbations may differ well from case to case, depending on the initial conditions of the problem.

Indeed, let us assume that the ratio of the mean motions of a minor planet and of Jupiter is close to that of two small coprime integers, say

$$\frac{n}{n_J} \approx \frac{p+q}{p} \ . \tag{49}$$

Then the most sizable long-range perturbations of the minor planet will originate from those terms in the disturbing function (48) that have indices

$$j = -hq \ , \quad k = h(p+q) \ , \qquad (h = 0, \pm 1, \pm 2, \dots) \ .$$

The corresponding arguments are

$$jM + k\psi = (j+k)M + k(\omega - n_J t) = h\Theta \ ,$$

where

$$\Theta = pM + (p+q)(\omega - n_J t) \ .$$

In the classical process of formal integration such terms give rise to secular ($h = 0$) and long-periodic ($h \neq 0$) perturbations of the elements, the latter being large because of the small divisors $h[pn-(p+q)n_J]$, the squares of which appear in the perturbations of the mean anomaly M. The size of the long-periodic perturbations also depends on the rank q of the commensurability, because the coefficients C_{jk} in the development (48) contain $e^{|j|}$ as a factor, and $|j| = |h|q$ in the critical terms.

This type of small-divisor problem is still challenging the mathematician and is of great interest to the astronomer, for it bears on the structure of our solar system. While numerous minor planets have mean motions nearly commensurable with that of Jupiter, they seem to shun certain regions of commensurability. The distribution of minor planets according to their mean motions is thus conspicuously uneven. Similar conditions exist for some satellites of the major planets, and also for the ring of Saturn. A detailed description of the prevailing situation and ample references to the literature of the subject were given recently in Hagihara's (1961) report.

Notwithstanding the inherent mathematical difficulties, we can gain valuable insight into the nature of nearly commensurate motion as follows. By what is traditionally called the method of Delaunay (1860, 1867), in each particular case we consider only the critical terms of the disturbing function, which therefore simplifies to

$$\bar{R} = C_o + 2 \sum_{1}^{\infty} C_h \cos h\Theta , \quad \text{with} \quad C_h = C_{-hq,\, h(p+q)} . \qquad (50)$$

It is quite plausible then that the solutions to the "averaged" dynamical problem defined by equations (46) and $F = 1/2L^2 + \bar{R}$ will exhibit the essential properties of the long-range perturbations in the original problem belonging to the complete disturbing function (48). In principle, the integration of the averaged restricted problem of three bodies can be reduced to quadratures. But this is not a practical procedure; one preferably resorts to numerical integration, a device available in much more complicated cases.

As to the justification of Delaunay's method, generalized by Poincaré (1893) and von Zeipel (1916), we merely remark that since the middle thirties and under slightly different circumstances it has received special attention in the modern theory of nonlinear oscillations. Let us refer to the work of Bogoliubov and Mitropolsky (1958). The concern here is with the treatment of the canonical system (46).

Following Poincaré (1902) and Andoyer (1903) we perform two simple canonical transformations. In order to eliminate the explicit dependence on time of the Hamiltonian F, we introduce first the angular variable

$$\chi = \omega - n_J t \quad ,$$

whereupon the system (46) becomes,

$$\dot{L} = H_M , \quad \dot{G} = H_\chi , \quad \dot{M} = -H_L , \quad \dot{\chi} = -H_G , \tag{51}$$

the new Hamiltonian being

$$H = F + n_J G = 1/2L^2 + n_J G + R \quad ,$$

and

$$R = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} C_{jk}(L, G) \cos [(j+k)M + k\chi] \quad . \tag{52}$$

Then $H = \text{const}$ is the well-known integral of Jacobi in the restricted problem of three bodies. A second canonical transformation

$$\begin{aligned} \Lambda &= (1 + \frac{p}{q}) L - \frac{p}{q} G & \psi &= M + \chi \\ \Gamma &= - L + G & \Theta &= \frac{p}{q} M + (1 + \frac{p}{q}) \chi \quad , \end{aligned} \tag{53}$$

with the inverse

$$\begin{aligned} L &= \Lambda + \frac{p}{q}\Gamma & M &= (1 + \frac{p}{q}) \psi - \Theta \\ G &= \Lambda + (1 + \frac{p}{q})\Gamma & \chi &= - \frac{p}{q}\psi + \Theta \end{aligned} \tag{54}$$

makes the submultiple $\Theta = \circledcirc/q$ of the critical argument one of the angular variables and eliminates the inconvenience of using the Delaunay variable G for the description of moderately eccentric orbits.* The geometrical meaning of the variables Λ and Γ is easier to visualize if we represent them in the form

* The special case $p = 0$, $q = 1$, although meaningless in relation to commensurabilities, is important in the theory of general perturbations. Note that $(-2\Gamma)^{1/2} \sin \Theta$, $(-2\Gamma)^{1/2} \cos \Theta$ is also a pair of canonical elements.

$$\Lambda = \alpha^{\frac{1}{2}}, \quad \Gamma = -\alpha^{\frac{1}{2}} \frac{\epsilon^2}{2}, \tag{55}$$

because the quantities

$$\alpha = \Lambda^2, \quad \epsilon = \left(-\frac{2\Gamma}{\Lambda}\right)^{\frac{1}{2}} \tag{56}$$

are analogous to the conventional semimajor axis and eccentricity of an ellipse. Really, comparison of the definitions (45), (54) and (55) yields the relations

$$a = \alpha(1 - \varkappa\epsilon^2)^2, \quad (1 - e^2)^{\frac{1}{2}} = 1 - \frac{\epsilon^2}{2(1 - \varkappa\epsilon^2)}, \quad e = \frac{\epsilon\left[1 - (\varkappa + \frac{1}{4})\epsilon^2\right]^{\frac{1}{2}}}{1 - \varkappa\epsilon^2}, \tag{57}$$

where we put $\varkappa = \frac{p}{2q}$. The quantity $\nu = \alpha^{-\frac{3}{2}} = \Lambda^{-3} = n(1 - \varkappa\epsilon^2)^3$ is analogous to the conventional mean motion. Expanding the coefficients C_{jk} of the disturbing function in terms of the canonical elements Λ, Γ is clearly equivalent to that in terms of the auxiliary quantities α, ϵ. We will show how the latter development can be carried out with the use of generalized Newcomb operators.

Now consider the averaged dynamical problem

$$\dot{\Lambda} = H_\psi, \quad \dot{\Gamma} = H_\Theta, \quad \dot{\psi} = -H_\Lambda, \quad \dot{\Theta} = -H_\Gamma,$$

where

$$H = \tfrac{1}{2}\left(\Lambda + \frac{p}{q}\Gamma\right)^{-2} + n_J\left[\Lambda + \left(1 + \frac{p}{q}\right)\Gamma\right] + \overline{R},$$

and $\overline{R}$ was given by equation (50). Since the Hamiltonian H does not depend on the angular variable ψ, besides H = const, we have here the integral Λ = const. Therefore it is sufficient to treat the reduced canonical system

$$\dot{\Gamma} = \overline{R}_\Theta, \quad \dot{\Theta} = \left[\frac{p}{q}n - \left(1 + \frac{p}{q}\right)n_J\right] - \overline{R}_\Gamma; \tag{58}$$

when this system has been solved, the function $\psi(t)$ follows by integrating the equation

$$\dot{\psi} = (n - n_J) - \overline{R}_\Lambda. \tag{59}$$

A detailed numerical investigation of the canonical system (58), (59) for various cases of commensurabilities is planned at the SAO.

Our present problem is to produce the coefficients C_h in equation (50) as power series of the auxiliary eccentricity ϵ. Once again we begin with the familiar development

$$
\begin{aligned}
R_0 &= m_J\{(1+\alpha^2-2\alpha\cos\psi)^{-\frac{1}{2}}-\alpha\cos\psi\} = \sum_{-\infty}^{\infty} B_k(\alpha)\xi^k ,\\
B_1(\alpha) &= B_{-1}(\alpha) = m_J[b_1^{\frac{1}{2}}(\alpha)-\tfrac{1}{2}\alpha] ,\\
B_k(\alpha) &= m_J\, b_k^{\frac{1}{2}}(\alpha) \qquad \text{for all other indices,}
\end{aligned}
\tag{60}
$$

pertaining to circular orbits. By direct application of the principles elaborated in part 1, the symbolic development of the disturbing function (47) in terms of the variables α, ϵ, z, ξ is

$$
R = \sum_{-\infty}^{\infty} (r/\alpha)^D (x/z)^k B_k \xi^k , \quad \text{where } D = \alpha\partial/\partial\alpha .
$$

Anew we can set

$$
(r/\alpha)^D (x/z)^k = \sum_{-\infty}^{\infty}\left\{\sum_m \prod\nolimits_j^m (D|k)\,\epsilon^m\right\} z^j ,
$$

but the expressions of the Newcomb operators $\prod_j^m (D|k)$ are different from the former ones. In analogy with equations (35), we define polynomial coefficients $\Xi_{\rho,\sigma}^{n,k}$ by the expansion

$$
(r/\alpha)^n (x/z)^k = \sum_0^{\infty}\sum_0^{\infty} \Xi_{\rho,\sigma}^{n,k}\, \epsilon^{\rho+\sigma} z^{\rho-\sigma} . \tag{61}
$$

Then

$$
\Xi_{\rho,\sigma}^{n,k} = \prod\nolimits_{\rho-\sigma}^{\rho+\sigma}(n|k) ,
$$

and the generalized Newcomb operators are

$$
\prod\nolimits_j^m (D|k) = \Xi_{\frac{m+j}{2},\frac{m-j}{2}}^{D,k} . \tag{62}
$$

It is altogether possible to find recurrence relations satisfied by the polynomials $\Xi_{\rho,\sigma}^{n,k}$, similar to, but much more complicated than equations (38) and (39). An easier approach to their construction consists in the following. We assume that the polynomials $X^{n,k}_{\rho,\sigma}$ relating to the conventional eccentricity have already been obtained. The last of equations (57) defines e as an analytic function of the auxiliary eccentricity ϵ, and generates the coefficients $a_{\ell,\lambda}$ in the power series

$$e^{\ell} = \sum_{0}^{\infty} a_{\ell,\lambda}\, \epsilon^{\ell+2\lambda} , \tag{63}$$

which are rational numbers depending on the adopted value of $\varkappa$. For any exponent $\ell \geq 0$ we have

$$a_{\ell,0} = 1 ; \quad \text{moreover} \quad a_{0,\lambda} = 0, \ \text{if} \ \lambda = 1, 2, \ldots .$$

The identity

$$\sum_{0}^{\infty} a_{1,\mu}\, \epsilon^{1+2\mu} \sum_{0}^{\infty} a_{\ell,\nu}\, \epsilon^{\ell+2\nu} = \sum_{0}^{\infty} a_{\ell+1,\lambda}\, \epsilon^{\ell+1+2\lambda}$$

furnishes the recurrence relations

$$a_{\ell+1,\lambda} = \sum_{\mu=0}^{\lambda} a_{1,\mu}\, a_{\ell,\lambda-\mu} , \tag{64}$$

and the coefficients

$$a_{2,0} = 1 , \quad a_{2,\lambda} = (\varkappa - \lambda/4)\varkappa^{\lambda-1}$$

are known. So the equations

$$\sum_{\mu=0}^{\lambda} a_{1,\mu}\, a_{1,\lambda-\mu} = (\varkappa-\lambda/4)\varkappa^{\lambda-1}$$

determine the coefficients

$$\begin{aligned}
a_{1,1} &= \varkappa/2 - 1/8 , \\
a_{1,2} &= 3\varkappa^2/8 - 3\varkappa/16 - 1/128 , \\
a_{1,3} &= 5\varkappa^3/16 - 15\varkappa^2/64 - 5\varkappa/256 - 1/1024 , \\
a_{1,4} &= 35\varkappa^4/128 - 35\varkappa^3/128 - 35\varkappa^2/1024 - 7\varkappa/2048 - 5/32768 , \ \ldots ,
\end{aligned}$$

in succession, and for $\ell \geq 3$ one can use the relations (64).

Having obtained the expansion (63) of e^{ℓ} into powers of ε, we define the intermediary polynomials $\xi^{n,k}_{\beta,\gamma}$ by the rearrangement

$$(r/a)^n (x/z)^k = \sum_0^\infty \sum_0^\infty X^{n,k}_{\rho,\sigma} [e(\varepsilon)]^{\rho+\sigma} z^{\rho-\sigma}$$

$$= \sum_0^\infty \sum_0^\infty X^{n,k}_{\rho,\sigma} \left\{ \sum_0^\infty a_{\rho+\sigma,\lambda}\, \varepsilon^{\rho+\sigma+2\lambda} \right\} z^{\rho-\sigma} = \sum_0^\infty \sum_0^\infty \xi^{n,k}_{\beta,\gamma}\, \varepsilon^{\beta+\gamma} z^{\beta-\gamma} ,$$

which gives

$$\xi^{n,k}_{\beta,\gamma} = \sum_{\lambda} a_{\beta+\gamma-2\lambda,\lambda} X^{n,k}_{\beta-\lambda,\gamma-\lambda} , \quad (\lambda = 0, 1, \ldots, \min(\beta, \gamma)) . \qquad (65)$$

On the other hand

$$(r/\alpha)^n (x/z)^k = (a/\alpha)^n (r/a)^n (x/z)^k ,$$

and according to the first of equations (57) we can write

$$(a/\alpha)^n = (1-\varkappa\varepsilon^2)^{2n} = \sum_0^\infty (-\varkappa)^\tau \binom{2n}{\tau} \varepsilon^{2\tau} .$$

Thus the expansion (61) becomes

$$(r/\alpha)^n (x/z)^k = \sum_0^\infty \sum_0^\infty \sum_0^\infty (-\varkappa)^\tau \binom{2n}{\tau} \xi^{n,k}_{\beta,\gamma}\, \varepsilon^{\beta+\gamma+2\tau} z^{\beta-\gamma} ,$$

that is

$$\Xi^{n,k}_{\rho,\sigma} = \sum_{\tau} (-\varkappa)^\tau \binom{2n}{\tau} \xi^{n,k}_{\rho-\tau,\sigma-\tau} , \quad (\tau = 0, 1, \ldots, \min(\rho, \sigma)) . \qquad (66)$$

In conclusion, we summarize the formulas relating to the canonical development of the disturbing function in the planetary restricted problem of three bodies, and to the computation of the coefficients C_h in particular. The complete disturbing function is

$$R = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} C_{jk} \cos [(j+k)M + k\chi] , \qquad (52)$$

with $$C_{jk} = \sum_m C_{jk}^m \, \epsilon^m \quad , \quad (m - |j| = 0, 2, 4, \ldots) \; ,$$

and $$C_{jk}^m = \prod{}_j^m (D|k) B_k \; ;$$

here the generalized Newcomb operators $\prod{}_j^m(D|k)$ and the functions B_k are defined by equations (62), (66), (65), and (60). Let the case of near commensurability under consideration be characterized by the condition (49). Then the critical part of the disturbing function is

$$\bar{R} = C_o + 2\sum_1^\infty C_h \cos hq\Theta \, , \tag{50}$$

where $$C_h = C_{-hq, h(p+q)} \, , \quad \text{and} \quad \Theta = \frac{p}{q} M + \left(1 + \frac{p}{q}\right)\chi \; .$$

Writing

$$C_h = \sum_m C_h^m \, \epsilon^m \quad , \quad (m - hq = 0, 2, 4, \ldots)$$

we have

$$C_h^m = \prod{}_{-hq}^m (D|h(p+q)) \, B_{h(p+q)} \quad , \tag{67}$$

and

$$\prod{}_{-hq}^m (D|h(p+q)) = \boxminus^{D, h(p+q)}_{\frac{m-hq}{2}, \frac{m+hq}{2}} = \boxminus^{D, -h(p+q)}_{\frac{m+hq}{2}, \frac{m-hq}{2}} \; .$$

3. Programming methods

3.1. Introduction. -- As mentioned in the Introduction to this report, an IBM 709 computer has been used to generate the polynomials $J_{\rho,\sigma}^{n,k}$ and $\Xi_{\rho,\sigma}^{n,k}$ that are defined in equations (40) and (61), for a range of values of the indices ρ and σ. This part of the report describes the computer programs that were employed. These programs enabled the computer to perform tedious algebra, albeit of a rather restricted nature. It is certain that computers have been applied since their advent to comparable algebraic problems that were of concern in individual scientific studies. These applications have been rather few in number and have had very scant reporting in the literature. Perhaps the earliest was the use of EDSAC I to generate representations of formulas for molecular integrals that were needed in some calculations of theoretical chemistry (Boys et al., 1956). Boys dealt with formulas that were represented conveniently by arrays of numerical coefficients. Several later algebraic applications of computers, including that described here, come into the same general category. Subroutines that manipulate arrays of coefficients representing polynomials in one or more variables have been coded in several laboratories. Increasing attention is now being given to more symbolic processes. Demonstrations have been given of the ability of various "symbol manipulating" programs to generate formulas in accordance with simple algorithms. Significant applications of such programs to physical problems have received little mention, however, in the scientific literature. The need to manipulate literal algebraic and analytical expressions in theoretical chemical studies has led one of us (M. P. B.) to the development of some formalisms and programs for symbol manipulation whose application to mechanized algebra is now under investigation in the Cooperative Computing Laboratory.

The aspect of the Newcomb operator work that we wish to stress most is the utter simplicity of the programming techniques that were used. It seems likely that programs of equal simplicity could deal with significant algebraic problems in other fields, for which this possibility has not yet been explored. Essentially, the $J_{\rho,\sigma}^{n,k}$ and $\Xi_{\rho,\sigma}^{n,k}$ polynomials were represented in the computer by arrays of coefficients. Trivial subroutines were coded to add, subtract, and multiply polynomials in this representation. These basic subroutines were called by further subroutines, which embodied the recurrence procedures defined mathematically in the earlier parts of this report. The sequential nature of mechanized computing requires the production of results in some definite sequence. The sequence that is appropriate to a particular problem defines the over-all logical structure, that is the major loops and branches, in the program used. The sequence in which it is convenient to generate the $J^{n,k}_{\rho,\sigma}$ and $\Xi^{n,k}_{\rho,\sigma}$ polynomials determines the general structure of our programs. The recurrence formulas themselves determine simple sequences of coding that call the basic polynomial manipulating subroutines.

The programs that generate the coefficients in the $J^{n,k}_{\rho,\sigma}$, $\Xi'^{n,k}_{\rho,\sigma}$ and other polynomials were coded in the FORTRAN II language. The programs construct polynomials for values of ρ and σ such that $\rho \geq \sigma$, and $\rho + \sigma$ takes successive integer values from 0 to a limit provided in the input data. When the $\Xi^{n,k}_{\rho,\sigma}$ polynomials are constructed, the relevant values of $\varkappa$ must also be provided in the input. The precise format of the input is under the control of a very simple portion of the program, which can be adapted to different operating circumstances without affecting the subroutines that incorporate the mathematical procedures. The output format can also be adapted to different requirements by changing some output statements and output subroutines in a simple manner. The internal representation of the polynomials is described in part 3.2. The basic polynomial manipulating subroutines are described in part 3.3. The higher-level subroutines that produce the $J^{n,k}_{\rho,\sigma}$ and $\Xi^{n,k}_{\rho,\sigma}$ polynomials are described in part 3.4. The results were printed in the manner conventionally used for computer output. Some of the results were also transformed by further programs into the code of the Photon S-560 photographic typesetting unit. This unit and its applications to the printing of computer results are described elsewhere (Barnett, Moss, Luce and Kelley, 1963). A short program was used to transform the initial computer output to a form that could be processed by the CARDPRINT system to produce the codes that drive the Photon machine. These codes were punched on paper tape, and the tape was used to drive the S-560 unit in the Cooperative Computing Laboratory. The film that was exposed in the Photon unit to the images of the appropriate sequences of characters, under the control of the paper tape, was developed and used as the original from which part 4 of this report was copied photolithographically.

The coefficients in the $J^{n,k}_{\rho,\sigma}$ polynomials are integers, those in the $\Xi^{n,k}_{\rho,\sigma}$ polynomials are rational numbers. The version of the programs that is in use at present represents all coefficients as double precision floating point numbers throughout, to avoid excessive roundoff. If so desired the coefficients in the $J^{n,k}_{\rho,\sigma}$ may be converted from this form to integer representation for printed output. The coefficients in the $\Xi^{n,k}_{\rho,\sigma}$ are recorded in single precision floating point form. The only details of the program that are specific to the double precision nature of the operations are the limits on the sizes of array that can be used to store the coefficients, and the D codes in the arithmetic statements.

3.2. Internal representation. -- Similar storage conventions are used for the $J^{n,k}_{\rho,\sigma}$, $X^{n,k}_{\rho,\sigma}$, $\xi^{n,k}_{\rho,\sigma}$ and $\Xi^{n,k}_{\rho,\sigma}$ polynomials. These may be explained by reference to the $J^{n,k}_{\rho,\sigma}$ polynomials. The subscripts ρ and σ are stored as integer variables MA and MB. The mnemonics used in the present version of the programs correspond to versions of the mathematical notation that preceded those described in parts 1 and 2 of this report, but this should cause no confusion. The coefficients of the polynomial $J^{n,k}_{MA,MB}$ are stored in consecutive locations of a double precision array AICM of dimension 3750, starting with the word AICM(N), whose index N is stored for reference in another array IN. This index is called the pointer to $J^{n,k}_{MA,MB}$. The array IN is of dimension 15×15, and the pointer to $J^{n,k}_{MA,MB}$ is stored in IN(MA + 1, MB + 1). The word AICM(N) contains g + 1, where g is the order in n of $J^{n,k}_{MA,MB}$. The word AICM(N + 1) contains h + 1, where h is the order of the polynomial in k that is independent of n. The coefficients of increasing powers of k in this polynomial are stored as double precision numbers in successive locations of AICM. The polynomial in k that multiplies n is stored next, in a similar fashion; then the polynomial in k that multiplies n^2 and so on, up to the polynomial in k that multiplies n^g.

As an example, consider the polynomial

$$J^{n,k}_{2,1} = (-2k - 10k^2 - 8k^3) + (3 + 5k + 4k^2)n + (1 + 2k)n^2 - n^3 .$$

For reasons that are explained shortly, this is stored with its pointer N equal to 43. Accordingly, IN(2 + 1, 1 + 1) = 43, and the contents of AICM(43) *et seq.* are as follows:

INX	AICM(INX)
43	4
44	4
45	0
46	- 2
47	- 10

48	- 8
49	3
50	3
51	5
52	4
53	2
54	1
55	2
56	1
57	- 1

The mnemonic INX is used above to index the AICM array. This array is used to store all the computed polynomials. A similar array, ATEMP, is used to store the $\xi^{n,k}_{\rho,\sigma}$ and $\Xi^{n,k}_{\rho,\sigma}$ polynomials as they are computed. During the execution of the program it is necessary to operate on the polynomials that have been formed and stored. Whenever a polynomial is to be used, it is first transferred from AICM into one of four temporary stores AICT 1, AICT 2, AICT 3, AICT 4. These temporary stores are 16×16 arrays in which the double precision element with subscripts (I, J) is the coefficient of $n^{I-1}\, k^{J-1}$ in the polynomial , the element with subscripts (I, 16) is one greater than the order in k of the polynomial coefficient of n^{I-1} and the element with subscripts (16, 16) is one greater than the order of the entire polynomial in n.

Thus if the polynomial $J^{n,k}_{2,1}$ is transferred to AICT 1, the contents of this array are as follows :

AICT (I, J)

		J→						
		1	2	3	4	5	...	16
I	1	0	-2	-10	-8			4
↓	2	3	5	4				3
	3	1	2					2
	4	-1						1
	.							
	.							
	.							
	16							4

3.3. Basic polynomial manipulation subroutines. -- Several subroutines are used to operate on polynomials whose representations are stored in the 16×16 arrays AICT 1, AICT 2, AICT 3, AICT 4. These are described by reference to the case where AICT 1 and AICT 2 are the arguments in the calling sequence, but any of the four arrays (or any equivalent array) can appear where AICT 1 and AICT 2 are used below.

CLEAR(AICT 1) This clears all words of AICT 1 to zero.

ADD(AICT 1, AICT 2). This adds the polynomials whose representations are stored in AICT 1 and AICT 2, and stores the representation of the result in AICT 2.

MPCONS(AMU, AICT 1). This multiplies the polynomial whose representation is stored in AICT 1 by a constant, stored in AMU, and stores the representation of the resulting polynomial in AICT 1.

MPMC(AMU, AICT 1, AICT 2). This performs the same operation as MPCONS, but stores the result in AICT 2, leaving the contents of AICT 1 unchanged.

MPMK(AMU, AICT 1, AICT 2). This multiplies the polynomial whose representation is stored in AICT 1 by the linear factor ak, and stores the representation of the result in AICT 2. The letter a is used here to denote the numerical value of a constant, stored in AMU. The second and third arguments of this subroutine must not be the same.

MPMN(AMU, AICT 1, AICT 2). This multiplies the polynomial whose representation is stored in AICT 1 by the linear factor an, and stores the representation of the result in AICT 2. The letter a is used as it is in the explanation of MPMK. The second and third arguments of this subroutine must not be the same.

SUBST(AMU, AICT 1). This substitutes the linear factor $(k + a)$ for k in the polynomial whose representation is stored in AICT 1. The letter a is used as in the preceding paragraphs. This subroutine uses the binomial coefficients stored in the COMMON array APASC.

PASCT(APASC). This forms Pascal's triangle in the 15×15 COMMON array APASC.

TRANS(N, AICT 1). This transfers the polynomial whose pointer is N from the COMMON array AICM to the working array AICT 1. The nature of AICM and N was explained in part 3.3. The value of N is unchanged by the subroutine.

STORE(N, AICT 1). This copies the polynomial whose representation is stored in AICT 1, into the AICM array, starting at AICM(N). The value of N is reset by the subroutine to the index of the word in AICM immediately following the last of the words required for the polynomial. Before exit the subroutine calls the subroutine PRINT (see below). It is used to store the $J^{n,k}_{\rho,\sigma}$ and $X^{n,k}_{\rho,\sigma}$ polynomials in AICM.

KEEP(N, AICT 1). This differs from STORE only in that it transfers the polynomial from the array AICT 1 to the array ATEMP. It is used to store the $\xi^{n,k}_{\rho,\sigma}$ and $\Xi^{n,k}_{\rho,\sigma}$ polynomials.

PRINT(AICT 1). This prints the contents of AICT 1 if the data card so requests. An earlier version of the subroutine printed intermediate results for program testing and correcting. The present version includes identifying data in the record (polynomial name and order).

DIVIDE(MA, MB, AICT 1). This divides the polynomial whose representation is stored in AICT 1 by 2^{MA+MB} MA! MB! It is used to convert $J^{n,k}_{\rho,\sigma}$ to $X^{n,k}_{\rho,\sigma}$.

The internal operation of these subroutines is very simple. Most of them use the variables NORDP 1, KORDP 1, MNP 1 and MKP 1 in a uniform manner. These variables are defined independently in each of the subroutines that use them. The variable NORDP 1 is set equal to a number that is one greater than the order in n of the polynomial on which an operation is being performed. The variable MNP 1 is a running index that at any time is one greater than the power of n under attention. It takes values 1 to NORDP 1 in turn. The variable KORDP 1 is set equal to a number one greater than the order in k of the polynomial coefficient of the power of n that is under attention. The variable MKP 1 is a running index that at any time is one greater than the power of k under attention. It takes values 1 to KORDP 1 in turn. The variables MNP 1 and MKP 1 are used as indices of two DO loops whose respective upper limits are NORDP 1 and KORDP 1. The different low level subroutines perform different operations within this simple nest of DO loops. The subroutine CLEAR sets NORDP 1 and KORDP 1 both equal to 16 to clear the entire array.

The subroutine PASCT constructs the binomial coefficients using

$$\binom{n}{0} = 1 , \quad \binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i} , \quad i > 0 .$$

The operation of SUBST corresponds to the summation that can be expressed as follows. Let the contents of AICT 1 represent the polynomial $c(n,k)$,

where $$c(n,k) = \sum_{i=o}^{g} \sum_{j=o}^{h_i} c_{i,j} n^i k^j .$$

Then

$$c(n,k+a) = \sum_{i=o}^{g} \sum_{j=o}^{h_i} c'_{i,j} n^i k^j ,$$

where

$$c'_{i,j} = c_{i,j} + \sum_{\ell=j+1}^{h_i} c_{i,\ell} a^{\ell-j} \binom{\ell}{j} .$$

In the subroutine SUBST, the array AITEMP stores powers of the argument AMU (i.e. a in the above equations).

3.4. <u>High-level programs.</u> -- The mathematical procedures developed in parts 1 and 2 of this report form the basis of five major subroutines. These are

GETJS ,	which constructs the $J^{n,k}_{\rho,\sigma}$ polynomials defined	in equation (40)
GETXS ,	" " " $X^{n,k}_{\rho,\sigma}$ " "	in equation (35)
GETAS ,	" " " $a_{\ell,\lambda}$ constants "	in equation (63)
GETPSI ,	" " " $\xi^{n,k}_{\rho,\sigma}$ polynomials "	in equation (65)
BIGPSI ,	" " " $\Xi^{n,k}_{\rho,\sigma}$ " "	in equation (61) .

A short main program is used to read data and to call this system of subroutines. A data card may request the calculation of the $X^{n,k}_{\rho,\sigma}$ polynomials up to some limiting value of $\rho+\sigma$. A data card may request the calculation of $\Xi^{n,k}_{\rho,\sigma}$ polynomials up to some limiting value of $\rho+\sigma$, and for some value of $\varkappa$. Depending on the circumstances, the main program calls the subroutine GETXS, or the subroutine BIGPSI, or both, before reading another data card. The subroutine GETXS calls the subroutine GETJS which forms and records the $J^{n,k}_{\rho,\sigma}$ polynomials. The subroutine GETXS then converts the $J^{n,k}_{\rho,\sigma}$ polynonomials to the $X^{n,k}_{\rho,\sigma}$ polynomials and records these. The subroutine BIGPSI calls the subroutine GETPSI to form the $\xi^{n,k}_{\rho,\sigma}$ polynomials. This in turn calls the subroutine GETAS to form the $a_{\ell,\lambda}$ coefficients necessary for generating the $\xi^{n,k}_{\rho,\sigma}$ polynomials from the $X^{n,k}_{\rho,\sigma}$ polynomials previously computed. Then BIGPSI uses the $\xi^{n,k}_{\rho,\sigma}$ polynomials to form and record the $\Xi^{n,k}_{\rho,\sigma}$ polynomials.

We will now describe the operations of the five major subroutines:

GETJS(AICM, MAPBMX). This subroutine generates representations of the polynomials $J^{n,k}_{\rho,\sigma}$ $0 \le \rho+\sigma \le$ MAPBMX, $\sigma \le \rho$, and stores these in the array AICM in the format which is described in part 3.2. The representation of the first polynomial is constructed from the explicit formula, taken from

$$J^{n,k}_{o,o} = 1 \ .$$

The recurrence formulas are then applied using the following mnemonics

MA = ρ , MB = σ , MAPB = $\rho + \sigma$,

MG = τ (in the $c_{\rho\sigma\tau}$ of equation (42)),

MGMAX = the maximum value of τ in the summation of equation (42),

CABG = $c_{\rho\sigma\tau}$ expressed as a floating point number,

MBMAX = the maximum value of σ for the value of MAPB under consideration.

The subroutine cycles around an outer loop in the index MAPB, which takes successive values 1 to MAPBMX. This loop starts at statement No. 50. In each cycle, the representation of $J^{n,k}_{\mathrm{MAPB},0}$ is constructed first, by a coding sequence that corresponds to equation (41), in which the second term is by-passed if MAPB = 1. An inner loop, which starts at statement No. 210, then cycles through values of MB from 1 to MBMAX, constructing representations of the $J^{n,k}_{\mathrm{MA},\mathrm{MB}}$ by a coding sequence that corresponds to equation (42). The second term is by-passed if MB $\le$ 1. The third term is by-passed if MA = 0. The summation over MG (that is τ) is omitted if MB = 1. When MB > 1, a DO loop with index MG, which starts at statement No. 315, constructs the sum

$$\sum_{\tau=2}^{\sigma} c_{\rho\sigma\tau} J^{n,k}_{\rho-\tau,\sigma-\tau}$$

The result is then multiplied by $\rho(\rho-\sigma+k)$. The coefficients $c_{\rho\sigma\tau}$ are evaluated by a coding sequence that corresponds to equation (43). Each term in the τ summation is constructed in AICT 1 and then added to AICT 2. Each $J^{n,k}_{\rho,\sigma}$ polynomial is formed in AICT 4, then printed and stored in AICM. The integer variable NNEXT is the running pointer to AICM; it is used as an argument of STORE.

GETXS(AICM, MAPBMX). This subroutine takes the representation of each $J^{n,k}_{\rho,\sigma}$ polynomial from the AICM array in turn, converts it to the representation of the corresponding $X^{n,k}_{\rho,\sigma}$, and overwrites the result in the AICM array from which the $J^{n,k}_{\rho,\sigma}$ was taken. The subroutine DIVIDE is used to perform the conversion, which simply entails division by $2^{\rho+\sigma}\rho!\,\sigma!$

GETAS(A, ANU, N). This subroutine constructs the $a_{\ell,\lambda}$ as elements of the two-dimensional array A. The subscripts I and J are used in part of the subroutine to identify elements of the array in which case $A(I,J) = a_{J,I-1}$. The subscripts in the program are in reverse order to those in the notation of part 2, as they correspond to an earlier version of the notation. The subscripts IP1 and L are used later in the subroutine in place of I and J. The argument ANU is the parameter $\varkappa$. When the subroutine is called by GETXS, the third argument is MAPBMX. Here, for conciseness, we denote it by N. The first part of the subroutine evaluates the integral part of (N/2), adds 1 and places the result in M. It then computes the A(1,J), A(I,1) and A(I,2), (that is the $a_{J,0}$, $a_{1,I-1}$ and $a_{2,I-1}$) as

$$A(1,J) = 1.0 \qquad \text{for} \quad J = 1 \text{ to } 2M$$

$$A(2,1) = \tfrac{1}{2}(\varkappa - \tfrac{1}{4}) \quad ,$$

then for I taking values 3 to M

$$A(I,1) = (\text{APROD} - \text{SUM})/2.0 \quad ,$$

where

$$\text{APROD} = (\text{ANU} - \frac{I-1}{4})\,\text{ANU}^{I-2}$$

The quantity SUM is evaluated by a loop on the index IH which takes values 2 to I-1. Each cycle of this loop adds A(IH, 1) A(I+1-IH, 1) to the accumulating value of SUM for the current I. This corresponds to the equation

$$a_{1,\lambda} = \tfrac{1}{2}(\varkappa - \lambda/4)\varkappa^{\lambda-1} - \tfrac{1}{2}\sum_{\mu=1}^{\lambda-1} a_{1,\mu}\, a_{1,\lambda-\mu} \ ,$$

which is obtained from the second equation after eq. (64) on page 24, by substituting 1 for $a_{1,0}$ and transposing.

Next, the elements A(I, 2) are constructed for I = 2 to M using the formula

$$A(I, 2) = (ANU - \frac{I-1}{4})ANU^{I-2} \ .$$

This corresponds to the equation on page 24,

$$a_{2,\lambda} = (\varkappa - \frac{\lambda}{4})\varkappa^{\lambda-1} \ .$$

The subroutine then cycles through a loop on the index L, which takes values from 3 to 2M-2; it is used as the second subscript of the A's. For each value of L, the limit IP1MAX of the first index is found from

$$IP1MAX = M - \text{integer part of}\left(\frac{L-1}{2}\right) \ .$$

The subroutine then cycles through a loop on the index IP1, which takes successive values 2 to IP1MAX. Within this loop A(IP1, L) is set equal to A(IP1, L-1), and then an inner loop on the index IHP1 is executed. This takes values 2 to IP1. Within this inner loop A(IP1, L) is increased by: A(IHP1, 1) × A(IP1 - IHP1 + 1, L-1). In this way A(IP1, L), i.e., $a_{L, IP1-1}$, is evaluated in accordance with equation (64).

GETPSI(AICM, MAPBMX, ANU). This subroutine uses the representations of the $X^{n,k}_{\rho,\sigma}$ polynomials in the array AICM, to generate representations of the $\xi^{n,k}_{\rho,\sigma}$ polynomials. It overwrites these in this same array immediately prior to exit. The $\xi^{n,k}_{\rho,\sigma}$ are found for $0 \le \rho + \sigma \le$ MAPBMX, and for $\varkappa$ = ANU. The subroutine calls GETAS to form the $a_{\ell,\lambda}$ in the array A. It then evaluates the $\xi^{n,k}_{\rho,\sigma}$ using equation (65) in the form

$$\xi^{n,k}_{\rho,\sigma} = X^{n,k}_{\rho,\sigma} + \sum_{\lambda=1}^{\sigma} a_{\rho+\sigma-2\lambda,\lambda}\, X^{n,k}_{\rho-\lambda,\sigma-\lambda} \; .$$

The limit on the summation in (65) is written as the lesser of the quantities ρ and σ. It can be written as σ here as the polynomials are constructed only for $\sigma \le \rho$.

The following mnemonics are used:

MA = ρ MB = σ MAPB = $\rho + \sigma$ MBMAX = maximum σ for current $\rho + \sigma$ value

I = λ J = $\rho + \sigma - 2\lambda$

AMULT = $a_{\rho+\sigma-2\lambda,\lambda}$ for current ρ, σ, λ

ATEMP = working array for $\xi^{n,k}_{\rho,\sigma}$.

The subroutine cycles through a major loop on the index MAPB which takes value 1 to MAPBMX. This loop starts at statement No. 50. Within this loop, the representation of $\xi^{n,k}_{\mathrm{MAPB},0}$ is formed first. An inner loop that starts at statement No. 210 is then executed with the index MB taking values 1 to MBMAX. This loop forms $\xi^{n,k}_{\mathrm{MA,MB}}$ by transferring the representation of $X^{n,k}_{\rho,\sigma}$ to the array AICT 2, and then executing the innermost loop on the index I which takes values 1 to MB. Each cycle of this innermost loop adds the product of A(I+1,J), (i.e., $a_{\rho+\sigma-2\lambda,\lambda}$) and the X polynomial with pointer IN(MA+1-I,MB+1-I), (i.e., $X^{n,k}_{\rho-\lambda,\sigma-\lambda}$) to the running sum. When the construction of a $\xi^{n,k}_{\rho,\sigma}$ is complete, it is recorded and transferred to the ATEMP array by the KEEP subroutine. This subroutine automatically increases the pointer NNEXT for use in storing the next $\xi^{n,k}_{\rho,\sigma}$. When all the $\xi^{n,k}_{\rho,\sigma}$ for $\rho + \sigma \le$ MAPBMX have been formed and stored in ATEMP, they are transferred to the AICM array by the loop that starts at statement No. 600.

BIGPSI(AICM, MAPBMX, ANU): This subroutine constructs representations of the polynomials $\Xi^{n,k}_{\rho,\sigma}$ for $0 \le \rho + \sigma \le$ MAPBMX, $\rho \ge \sigma$. The subroutine stores these in the array AICM prior to its conclusion. The argument ANU is the parameter $\varkappa$. The subroutine calls GETPSI to form the polynomials $\xi^{n,k}_{\rho,\sigma}$ in the array AICM. These are used by BIGPSI to form the polynomials $\Xi^{n,k}_{\rho,\sigma}$ one by one in the array AICT 2. Each $\Xi^{n,k}_{\rho,\sigma}$ is transferred to ATEMP after it is formed. When all the $\Xi^{n,k}_{\rho,\sigma}$ have been formed, they are transferred from the array ATEMP to the array AICM.

Each $\Xi^{n,k}_{\rho,\sigma}$ is constructed by use of a simple transformation of equation (66). Put

$$\varphi_t = -(2n - t + 1)\varkappa/t$$

then

$$\binom{2n}{\tau}(-\varkappa)^{\tau} = \varphi_{\tau}\,\varphi_{\tau-1}\cdots\varphi_1$$

and

$$\Xi^{n,k}_{\rho,\sigma} = (\ldots(((\xi^{n,k}_{\rho-\sigma,0})\times\varphi_{\sigma} + \xi_{\rho-\sigma+1,1})\times\varphi_{\sigma-1} + \xi_{\rho-\sigma+2,2})\times\varphi_{\sigma-2} + \ldots)$$
$$\times\varphi_1 + \xi^{n,k}_{\rho,\sigma} \quad .$$

The subroutine uses the following mnemonics

MA $= \rho$ MB $= \sigma$ MAPB $= \rho + \sigma$ MAMB $= \rho - \sigma$

MBMAX = maximum value of σ for current $\rho + \sigma$

N 1 = pointer to $\xi^{n,k}_{\rho-\sigma,0}$ in AICM

L = ℓ stored as an integer

BML = $\sigma - \ell$ stored as a floating point number

BMLP 1 = $\sigma - \ell + 1$ stored as a floating point number

MAMBPL = $\rho - \sigma + \ell$ stored as an integer

COEFF = $\varkappa/(\sigma - \ell + 1)$

N 2 = pointer to $\xi^{n,k}_{\rho-\sigma+\ell,\ell}$ in AICM .

The subroutine cycles through a major loop on MAPB, within which there is a loop on MB. Within this loop a polynomial $\Xi^{n,k}_{\rho,\sigma}$ is constructed. This is done by first transferring the representation of $\xi^{n,k}_{\rho-\sigma,0}$ to AICT 2. If MB = 0 (i.e. $\sigma = 0$), this is transferred to ATEMP, and the construction of the next $\Xi^{n,k}_{\rho,\sigma}$ is started. If MB > 0, the subroutine executes an inner loop on the index L. This takes values 1 to MB. In each cycle, the contents of AICT 2 is multiplied by COEFF, that is $\varkappa/(\sigma-\ell+1)$. In the first cycle, this is $\varkappa/\sigma$, in the last cycle it is $\varkappa$. The representation of the product of $-(2n - \sigma + \ell)$ and the polynomial represented in AICT 2 is then formed and stored in AICT 2. The polynomial $\xi^{n,k}_{\rho-\sigma+\ell,\ell}$ is then added. In the first cycle this is $\xi^{n,k}_{\rho-\sigma+1,1}$. In the last cycle it is $\xi^{n,k}_{\rho,\sigma}$. The loop thus corresponds to the equation on the top of page (36).

When the construction of a $\Xi^{n,k}_{\rho,\sigma}$ polynomial is complete, the subroutine KEEP records it and transfers it to the array ATEMP. The pointer NNEXT is reset automatically by KEEP for use in storing the next $\Xi^{n,k}_{\rho,\sigma}$. When all the $\Xi^{n,k}_{\rho,\sigma}$ for $\rho+\sigma \leq$ MAPBMX have been stored in ATEMP, they are transferred to AICM.

3.5. Output. -- The listings of a slight improvement of the programs which were used to construct the coefficients in the polynomials follow. The subroutine PRINT wrote the output on magnetic tape in three alternative forms, under the control of option requests in the data card and sense switch settings. The forms of output were: (i) bcd single precision floating point representations of the coefficients, with captions and in a format for printing on off-line IBM equipment. (ii) for the J's, the bcd integer representations of the coefficients, with captions and in a format for printing off-line, (ii) binary representations of the most significant word of each pair of words that were also stored in the AICM or ATEMP arrays. The subroutine INTVAL was used in the production of (ii), to convert the double precision floating point binary representation of an integer to its bcd integer representation. This subroutine used some utility subroutines of the BCD Manipulation Package (M.J. Bailey, P.B. Burleson, E.J.D. Carter, and K.L. Kelley, Cooperative Computing Laboratory Technical Note No.12, MIT 1963). For photocomposition, the output (iii) was read back into core storage by a program which produced the ρ code representation of the material to be printed (see M.P. Barnett, et al., 1963). This was written on a magnetic tape, which was then used as the input tape for the PC6 program. This wrote the requisite Photon codes on a magnetic tape, from which the paper tape to control the Photon machine was punched.

3.6 Program listings

```
CCLEAR
C       CLEAR TEMPORARY STORAGE
        SUBROUTINE CLEAR(AICT)
D       DIMENSION AICM(3750),APASC(15,15),ATEMP(3750),A(30,15)
        DIMENSION IPRINT(4),IN(15,15)
D       DIMENSION AICT(16,16)
        COMMON AICM,MA,MB,IJXPSI,IPRINT,APASC,ATEMP,A,IN,ANU
   10   DO 20 MNP1=1,16
        DO 20 MKP1=1,16
D 20    AICT(MNP1,MKP1)=0.
        RETURN
        END

CADD
C       ADD TWO POLYNOMIALS
        SUBROUTINE ADD(AICT1,AICT2)
D       DIMENSION AICM(3750),APASC(15,15),ATEMP(3750),A(30,15)
        DIMENSION IPRINT(4),IN(15,15)
D       DIMENSION AICT1(16,16),AICT2(16,16)
        COMMON AICM,MA,MB,IJXPSI,IPRINT,APASC,ATEMP,A,IN,ANU
D       NOP11=AICT1(16,16)
D       NOP12=AICT2(16,16)
        NOP1=XMAXOF(NOP11,NOP12)
D       AICT2(16,16)=NOP1
        DO 50 MNP1=1,NOP1
D       KOP11=AICT1(MNP1,16)
D       KOP12=AICT2(MNP1,16)
        KOP1=XMAXOF(KOP11,KOP12)
D       AICT2(MNP1,16)=KOP1
        DO 50 MKP1=1,KOP1
D 50    AICT2(MNP1,MKP1)=AICT1(MNP1,MKP1)+AICT2(MNP1,MKP1)
        RETURN
        END

CMPCONS
C       MULTIPLY BY CONSTANT
        SUBROUTINE MPCONS(ACONST,AICT)
D       DIMENSION AICM(3750),APASC(15,15),ATEMP(3750),A(30,15)
        DIMENSION IPRINT(4),IN(15,15)
D       DIMENSION AICT(16,16)
        COMMON AICM,MA,MB,IJXPSI,IPRINT,APASC,ATEMP,A,IN,ANU
D       NORDP1=AICT(16,16)
10      DO 20 MNP1=1,NORDP1
D       KORDP1=AICT(MNP1,16)
        DO 20 MKP1=1,KORDP1
D 20    AICT(MNP1,MKP1)=ACONST*AICT(MNP1,MKP1)
        RETURN
        END

CMPMC
C       MULTIPLY BY CONSTANT AND TRANSFER
        SUBROUTINE MPMC(AMU,AICT1,AICT2)
D       DIMENSION AICM(3750),APASC(15,15),ATEMP(3750),A(30,15)
        DIMENSION IPRINT(4),IN(15,15)
D       DIMENSION AICT1(16,16),AICT2(16,16)
        COMMON AICM,MA,MB,IJXPSI,IPRINT,APASC,ATEMP,A,IN,ANU
        DO 10 MNP1=1,16
        DO 10 MKP1=1,16
D 10    AICT2(MNP1,MKP1)=AICT1(MNP1,MKP1)
D       CALL MPCONS(AMU,AICT2)
        RETURN
        END

CMPMK
C       MULTIPLY BY K
        SUBROUTINE MPMK(AMU,AICT1,AICT2)
D       DIMENSION AICM(3750),APASC(15,15),ATEMP(3750),A(30,15)
        DIMENSION IPRINT(4),IN(15,15)
D       DIMENSION AICT1(16,16),AICT2(16,16)
        COMMON AICM,MA,MB,IJXPSI,IPRINT,APASC,ATEMP,A,IN,ANU
D       CALL CLEAR(AICT2)
D       NORDP1=AICT1(16,16)
D       AICT2(16,16)=NORDP1
        DO 50 MNP1=1,NORDP1
D       KORDP1=AICT1(MNP1,16)
D       AICT2(MNP1,16)=KORDP1+1
        DO 50 MKP1=1,KORDP1
D 50    AICT2(MNP1,MKP1+1)=AMU*AICT1(MNP1,MKP1)
        RETURN
        END

CMPMN
C       MULTIPLY BY N
        SUBROUTINE MPMN(AMU,AICT1,AICT2)
D       DIMENSION AICM(3750),APASC(15,15),ATEMP(3750),A(30,15)
        DIMENSION IPRINT(4),IN(15,15)
D       DIMENSION AICT1(16,16),AICT2(16,16)
        COMMON AICM,MA,MB,IJXPSI,IPRINT,APASC,ATEMP,A,IN,ANU
D       CALL CLEAR(AICT2)
D       NORDP1=AICT1(16,16)
D       AICT2(16,16)=NORDP1+1
        DO 50 MNP1=1,NORDP1
D       KORDP1=AICT1(MNP1,16)
D       AICT2(MNP1+1,16)=KORDP1
        DO 50 MKP1 = 1,KORDP1
D 50    AICT2(MNP1+1,MKP1)= AMU*AICT1(MNP1,MKP1)
        RETURN
        END
```

```
CSUBST
C       REPLACE K BY K+AMU
        SUBROUTINE SUBST(AMU,AICT)
D       DIMENSION AICM(3750),APASC(15,15),ATEMP(3750),A(30,15)
        DIMENSION IPRINT(4),IN(15,15)
D       DIMENSION AICT(16,16),AITEMP(15)
        COMMON AICM,MA,MB,IJXPSI,IPRINT,APASC,ATEMP,A,IN,ANU
D       IF(AMU)5,1000,5
D 5     AITEMP(1)=1.
        DO 10 I=2,15
D 10     AITEMP(I)=AITEMP(I-1)*AMU
D       NORDP1=AICT(16,16)
        DO 40 I=1,NORDP1
D       KORDP1=AICT(I,16)
        DO 40 J = 1,KORDP1
        IF(J-KORDP1)15,40,40
15      JP1=J+1
        DO 30 N=JP1,KORDP1
        MSUB=N-J+1
D 30    AICT(I,J)=AICT(I,J)+AICT(I,N)*APASC(N,J)*AITEMP(MSUB)
40      CONTINUE
1000    RETURN
        END

CPASCT
C       SUBROUTINE TO GENERATE
C        PASCAL'S  TRIANGLE
        SUBROUTINE PASCT(APASC)
D       DIMENSION AICM(3750),APASC(15,15),ATEMP(3750),A(30,15)
        DIMENSION IPRINT(4),IN(15,15)
        COMMON AICM,MA,MB,IJXPSI,IPRINT,APASC,ATEMP,A,IN,ANU
        DO 11 IIP1=1,15
D 11    APASC(IIP1,1)=1.
        DO 13 IIP1=2,15
D 13    APASC(IIP1,IIP1)=1.
        DO 20 IIP1=3,15
        IJP1MX=IIP1-1
        DO 19 IJP1=2,IJP1MX
D 19    APASC(IIP1,IJP1)=APASC(IIP1-1,IJP1-1)+APASC(IIP1-1,IJP1)
20      CONTINUE
        RETURN
        END

CTRANS
C       TRANSFER FROM MAIN TO TEMPORARY ARRAY
        SUBROUTINE TRANS(N,AICT)
D       DIMENSION AICM(3750),APASC(15,15),ATEMP(3750),A(30,15)
        DIMENSION IPRINT(4),IN(15,15)
D       DIMENSION AICT(16,16)
        COMMON AICM,MA,MB,IJXPSI,IPRINT,APASC,ATEMP,A,IN,ANU
D 10    CALL CLEAR(AICT)
        INX=N
D       NORDP1=AICM(INX)
D 20    AICT(16,16)=NORDP1
25      DO 50 MNP1=1,NORDP1
        INX=INX+1
D       KORDP1=AICM(INX)
D 30    AICT(MNP1,16)=KORDP1
        DO 50 MKP1=1,KORDP1
        INX=INX+1
D 50    AICT(MNP1,MKP1)=AICM(INX)
1000    RETURN
        END

CSTORE
C       TRANSFER FROM TEMPORARY TO MAIN ARRAY
        SUBROUTINE STORE(N,AICT)
D       DIMENSION AICM(3750),APASC(15,15),ATEMP(3750),A(30,15)
        DIMENSION IPRINT(4),IN(15,15)
D       DIMENSION AICT(16,16)
        COMMON AICM,MA,MB,IJXPSI,IPRINT,APASC,ATEMP,A,IN,ANU
D       NORDP1=AICT(16,16)
D       AICM(N)=NORDP1
        N=N+1
20      DO 100 MNP1=1,NORDP1
D       KORDP1=AICT(MNP1,16)
D       AICM(N)=KORDP1
        N=N+1
30      DO 100 MKP1=1,KORDP1
D       AICM(N)=AICT(MNP1,MKP1)
100     N=N+1
D       CALL PRINT(AICT)
        RETURN
        END
```

```
CKEEP
C          KEEPS GENERATED PSIS IN ATEMP
        SUBROUTINE KEEP(N,AICT)
D       DIMENSION AICM(3750),APASC(15,15),ATEMP(3750),A(30,15)
        DIMENSION IPRINT(4),IN(15,15)
D       DIMENSION AICT(16,16)
        COMMON AICM,MA,MB,IJXPSI,IPRINT,APASC,ATEMP,A,IN,ANU
D       NORDP1=AICT(16,16)
D       ATEMP(N)=NORDP1
        N=N+1
20      DO 100 MNP1=1,NORDP1
D       KORDP1=AICT(MNP1,16)
D       ATEMP(N)=KORDP1
        N=N+1
30      DO 100 MKP1=1,KORDP1
D       ATEMP(N)=AICT(MNP1,MKP1)
100     N=N+1
D       CALL PRINT(AICT)
        RETURN
        END

CPRINT
C       PRINT RESULTS
        SUBROUTINE PRINT(AICT)
D       DIMENSION AICM(3750),APASC(15,15),ATEMP(3750),A(30,15)
        DIMENSION IPRINT(4),IN(15,15)
D       DIMENSION AICT(16,16)
        DIMENSION MK(15),INTANS(16),INTEG(15,16)
        COMMON AICM,MA,MB,IJXPSI,IPRINT,APASC,ATEMP,A,IN,ANU
D 18    NORDP1=AICT(16,16)
        NORD=NORDP1-1
        IF(IJXPSI-2)15,10,15
   10   WRITE TAPE 12,AICT(16,16)
        DO 12 MNP1=1,NORDP1
D       KORDP1=AICT(MNP1,16)
D       WRITE TAPE 12,AICT(MNP1,16)
        DO 12 MKP1=1,KORDP1
D       WRITE TAPE 12,AICT(MNP1,MKP1)
   12   CONTINUE
   15   IF(IPRINT(IJXPSI))20,100,20
C            EXIT IF IPRINT(IJXPSI)=0
20      PRINT 21,NORD
  21    FORMAT(2H   //17H ALPHA PLUS BETA=I2)
        PRINT 262
262     FORMAT(8X,3HN,K)
263     GO TO (271,272,273,274),IJXPSI
271     PRINT 2711
2711    FORMAT(6X,6HJ     =)
        GO TO 30
272     PRINT 2722
2722    FORMAT(6X,6HX     =)


        GO TO 30
273     PRINT 2733
2733    FORMAT(4X,8HKSI     =)
        GO TO 30
274     PRINT 2744
2744    FORMAT(1X,11HBIGKSI     =)
30      PRINT 301,MA,MB
301     FORMAT(7X,I2,1H,I2)
   25   DO 50 MNP1=1,NORDP1
        MN=MNP1-1
D       KORDP1=AICT(MNP1,16)
        DO 26 I=1,KORDP1
26      MK(I)=I-1
        IF(IJXPSI-1) 27,27,28
   27   IF(SENSE SWITCH 6) 28,29
D       CALL INTVAL(FLTANS,INTANS)
   29   DO 127 MKP1=1,KORDP1
D       FLTANS=AICT(MNP1,MKP1)
        DO 128 K = 1,16
 128    INTEG(MKP1,K) = INTANS(K)
 127    CONTINUE
        PRINT 32,MN,((INTEG(MKP1,K),K=1,16),MK(MKP1),MKP1=1,KORDP1)
 32     FORMAT(/,7H     N**I2,8HTIMES    4(16A1,4H K**I2,2X),/17X,
      2 4(16A1,4H K**I2,2X),/17X,4(16A1,4H K**I2,2X),/17X,
      3 4(16A1,4H K**I2,2X))
        GOTO 50
 28     PRINT 31,MN,(AICT(MNP1,MKP1),MK(MKP1),MKP1=1,KORDP1)
 31     FORMAT(/,7H     N**I2,8HTIMES    4(E15.8,4H K**I2,2X),/17X,
      2 4(E15.8,4H K**I2,2X),/17X,4(E15.8,4H K**I2,2X),/17X,
      3 4(E15.8,4H K**I2,2X))
   50   CONTINUE
        IF(SENSE LIGHT 1) 990,1001
 1001   IF(IPRINT(IJXPSI))100,100,55
C            EXIT WITHOUT PUTTING COEFFS ONTO TAPE UNLESS IPRINT(IJXPSI)
C            IS GREATER THAN ZERO
D   55  WRITE TAPE 11,AICT(16,16)
        DO 57 MNP1=1,NORDP1
D       KORDP1=AICT(MNP1,16)
D       WRITE TAPE 11,AICT(MNP1,16)
        DO 57 MKP1=1,KORDP1
D       WRITE TAPE 11,AICT(MNP1,MKP1)
   57   CONTINUE
        GOTO 100
  990   PRINT 991,MA,MB
  991   FORMAT(37H INTEGER OUTPUT OVERFLOWS FOR ALPHA =I2,7H,BETA =I2,
      2 43H DEPRESS SENSE SWITCH 6 TO CONTINUE PROGRAM)
  992   IF(SENSE SWITCH 6) 18,992
  100   RETURN
        END
```

```
CINTVAL
C      CONVERT FLOATING TO INTEGER
       SUBROUTINE INTVAL(FLTANS,INTANS)
D      DIMENSION AICM(3750),APASC(15,15),ATEMP(3750),A(30,15)
       DIMENSION IPRINT(4),IN(15,15)
       DIMENSION INTANS(16)
       COMMON AICM,MA,MB,IJXPSI,IPRINT,APASC,ATEMP,A,IN,ANU
       CALL DEFBCD(1H ,INTANS(1))
D      COEFF=ABSF(FLTANS)+0.2
D      IF(10.**15-COEFF) 50,50,5
 50    DO 51 K=2,16
       CALL DEFBCD(1HX,INTANS(K))
 51    CONTINUE
       SENSE LIGHT 1
       GOTO 1000
 5     DO 100 K=2,16
D      INTANS(K) = COEFF/10.**(16-K)
       INTREM = INTANS(K)
D      COEFF = COEFF-FLOATF(INTREM)*10.**(16-K)
       IF (INTANS(K)) 80,80,120
 80    CALL DEFBCD(1H ,INTANS(K))
 100   CONTINUE
 2     CALL DEFBCD(7H0      +,INTANS(16))
       GOTO 1000
D 120  IF(FLTANS) 125,130,130
 125   CALL DEFBCD(1H-,INTANS(K-1))
       GOTO 135
 130   CALL DEFBCD(1H+,INTANS(K-1))
 135   INTANS(K)=IBCDFN(INTANS(K))
       CALL SFTLFT(INTANS(K),INTANS(K))
       IF(K-16) 140,1000,1000
 140   K=K+1
       DO 200 J=K,16
D      INTANS(J) = COEFF/10.**(16-J)
       INTREM = INTANS(J)
D      COEFF = COEFF-FLOATF(INTREM)*10.**(16-J)
       INTANS(J)=IBCDFN(INTANS(J))
       CALL SFTLFT(INTANS(J),INTANS(J))
  200  CONTINUE
 1000  RETURN
       END


CDIVIDE
C     DIVIDES BY (2**(MA+MB))*MA'*MB'
       SUBROUTINE DIVIDE(MA,MB,AICT)
D      DIMENSION AICM(3750),APASC(15,15),ATEMP(3750),A(30,15)
       DIMENSION IPRINT(4),IN(15,15)
D      DIMENSION AICT(16,16)
       COMMON AICM,MA,MB,IJXPSI,IPRINT,APASC,ATEMP,A,IN,ANU
D      AFACT=1.
D      BFACT=1.
       IF(MA)11,11,2
2      DO 10 M=1,MA
D 10   AFACT=AFACT*FLOATF(M)
 11    IF(MB)22,22,12
12     DO 20 M=1,MB
D 20   BFACT=BFACT*FLOATF(M)
D 22   DIV=2.**(MA+MB)*AFACT*BFACT
D      NORDP1=AICT(16,16)
       DO 30 MNP1=1,NORDP1
D      KORDP1=AICT(MNP1,16)
       DO 30 MKP1=1,KORDP1
D 30   AICT(MNP1,MKP1)=AICT(MNP1,MKP1)/DIV
       RETURN
       END
```

```
CGETJS
C      SUBROUTINE TO GENERATE J-COEFFICIENTS
       SUBROUTINE GETJS(AICM,MAPBMX)
D      DIMENSION AICM(3750),APASC(15,15),ATEMP(3750),A(30,15)
       DIMENSION IPRINT(4),IN(15,15)
D      DIMENSION AICT1(16,16),AICT2(16,16),AICT3(16,16),AICT4(16,16)
       COMMON AICM,MA,MB,IJXPSI,IPRINT,APASC,ATEMP,A,IN,ANU
D      CALL PASCT(APASC)
C                                  SET UP J(0,0)
5      IN(1,1)=1
D      AICM(1)=1.
D      AICM(2)=1.
D      AICM(3)=1.
       NNEXT=4
40     IJXPSI=1
50     DO 500 MAPB=1,MAPBMX
C                                  CONSTRUCT J(MA,0)
60     MA=MAPB
       MB=0
C                                  GET FIRST TERM
65     N1=IN(MA,1)
D 70   CALL TRANS(N1,AICT1)
D      CALL SUBST(1.,AICT1)
D      CALL MPMK(2.,AICT1,AICT4)
D      CALL MPMN(-1.,AICT1,AICT2)
D      CALL ADD(AICT2,AICT4)
80     IF(MA-1) 9999,120,90
C                                  GET SECOND TERM
90     N2=IN(MA-1,1)
D 95   CALL TRANS(N2,AICT1)
D      CALL SUBST(2.,AICT1)
D      CALL MPMK(1.,AICT1,AICT3)
D      CALL MPMN(-1.,AICT1,AICT2)
D      CALL ADD(AICT2,AICT3)
       IF(MA-2) 110,110,100
D 100  AMA1=MA-1
D      CALL MPCONS(AMA1,AICT3)
D 110  CALL ADD(AICT3,AICT4)
  120  IN(MA+1,1)=NNEXT
D      CALL STORE(NNEXT,AICT4)
C                                  CONSTRUCT J(MA,MB)
200    MBMAX=MAPB/2
       IF(MBMAX) 9999,500,210
210    DO 400 MB=1,MBMAX
       MAP1=MA
       MA= MA-1
C                                  GET FIRST TERM
  220  N1=IN(MAP1,MB)
D 225  CALL TRANS(N1,AICT1)
D      CALL SUBST(-1.,AICT1)


D      CALL MPMK(-2.,AICT1,AICT4)
D      CALL MPMN(-1.,AICT1,AICT2)
D      CALL ADD(AICT2,AICT4)
       IF(MB-1) 9999,250,230
C                                  GET SECOND TERM
  230  N2=IN(MAP1,MB-1)
D 235  CALL TRANS(N2,AICT1)
D      CALL SUBST(-2.,AICT1)
D      CALL MPMK(1.,AICT1,AICT3)
D      CALL MPMN(1.,AICT1,AICT2)
D      CALL ADD (AICT2,AICT3)
D      A1MB=1-MB
D      CALL MPCONS(A1MB,AICT3)
D      CALL ADD(AICT3,AICT4)
C                                  GET THIRD TERM
  250  IF (MA-1) 360,251,251
  251  N3=IN(MA,MB)
D 255  CALL TRANS(N3,AICT1)
D      AM5MB4=MA-5*MB+4
D      CALL MPMC(AM5MB4,AICT1,AICT3)
D      CALL MPMK(4.,AICT1,AICT2)
D      CALL ADD(AICT2,AICT3)
D      CALL MPMN(1.,AICT1,AICT2)
D      CALL ADD(AICT2,AICT3)
D      CMA =-MA
D      CALL MPCONS(CMA,AICT3)
D      CALL ADD(AICT3,AICT4)
C                                  GET GAMMA SUM
 300   MGMAX=MB
       IF(MGMAX-1)360,360,310
D 310  CALL CLEAR(AICT2)
315    DO 340 MG=2,MGMAX
       MAMGP1=MA-MG+1
       MBMGP1=MB-MG+1
       IF (MG-2)9999,320,325
D 320  CABG=3*(MA-1)*(MB-1)
       GO TO 330
D 325  CABG=(FLOATF(2*MAMGP1*MBMGP1*(2*MG-5))*CABG)/FLOATF(MG)
330    NG = IN(MAMGP1,MBMGP1)
D      CALL TRANS(NG,AICT1)
D      CALL MPCONS(CABG,AICT1)
D      CALL ADD(AICT1,AICT2)
340    CONTINUE
D      CMA=MA
D      CALL MPCONS(CMA,AICT2)
D      CALL MPMK(1.,AICT2,AICT1)
       IF(MA-MB)9999,355,350
```

```
D 350 SMAMB=MA-MB
D      CALL MPCONS(SMAMB,AICT2)
D      CALL ADD(AICT2,AICT4)
D 355 CALL ADD(AICT1,AICT4)
360    IN(MAP1,MB+1)=NNEXT
D      CALL STORE(NNEXT,AICT4)
400    CONTINUE
  500 CONTINUE
 9999 PRINT9900
9900   FORMAT(24H J POLYNOMIALS COMPLETED)
       CALL CLOCK(2)
       RETURN
       END
```

```
CGETXS
C      EVALUATE X-COEFFICIENTS
       SUBROUTINE GETXS(AICM,MAPBMX)
D      DIMENSION AICM(3750),APASC(15,15),ATEMP(3750),A(30,15)
       DIMENSION IPRINT(4),IN(15,15)
D      DIMENSION AICT1(16,16)
       COMMON AICM,MA,MB,IJXPSI,IPRINT,APASC,ATEMP,A,IN,ANU
D      CALL GETJS(AICM,MAPBMX)
       N=4
40     IJXPSI=2
50     DO 500 MAPB=1,MAPBMX
       MA=MAPB+1
       MBMX=MAPB/2
       DO 400 MB=0,MBMX
       MA = MA-1
D      CALL TRANS(N,AICT1)
D      CALL DIVIDE(MA,MB,AICT1)
D      CALL STORE(N,AICT1)
       IF(MB-MBMX)400,500,9999
400    IN(MA,MB+2)=N
500    IN(MAPB+2,1)=N
9999   PRINT 9998
9998   FORMAT(13H  END  OF  XS)
       CALL CLOCK(2)
       RETURN
       END
```

```
CGETAS
       SUBROUTINE GETAS(A,ANU,N)
D      DIMENSION AICM(3750),APASC(15,15),ATEMP(3750),A(30,15)
       DIMENSION IPRINT(4),IN(15,15)
       COMMON AICM,MA,MB,IJXPSI,IPRINT,APASC,ATEMP,A,IN,ANU
       M=1+N/2
       JA=2*M
       I=1
       DO 2 J=1,JA
D 2    A(I,J)=1.
C                                           COMPUTE FIRST ROW
       I=2
       J1=1
D      APROD=ANU-.25
D      A(I,J1)=APROD/2.
       DO 4 I=3,M
D      SUM=0.
D      AI=I
D      APROD=(ANU-(AI-1.)/4.)*(ANU**(I-2))
       IM1=I-1
       IP1=I+1
       DO 3 IH=2,IM1
       IPMH=IP1-IH
D 3    SUM=SUM+A(IH,J1)*A(IPMH,J1)
D  4   A(I,J1)=(APROD-SUM)/2.
C                                           COMPUTE SECOND ROW
       J2=2
       DO 5 I=2,M
D      AI=I
D 5    A(I,J2)=(ANU-(AI-1.)/4.)*(ANU**(I-2))
C                                           COMPUTE OTHER ROWS
       LMAX=2*M-2
       DO 10 L=3,LMAX
       IP1MAX=M-(L-1)/2
       DO 10 IP1=2,IP1MAX
D      A(IP1,L)=A(IP1,L-1)
       DO 10 IHP1=2,IP1
       INX1=IP1-IHP1+1
D 10   A(IP1,L)=A(IP1,L)+A(IHP1,1)*A(INX1,L-1)
       RETURN
       END
```

```
CGETPSI
C        GENERATES LITTLE PSIS
       SUBROUTINE GETPSI(AICM,MAPBMX,ANU)
D      DIMENSION AICM(3750),APASC(15,15),ATEMP(3750),A(30,15)
       DIMENSION IPRINT(4),IN(15,15)
D      DIMENSION AICT1(16,16),AICT2(16,16)
       COMMON AICM,MA,MB,IJXPSI,IPRINT,APASC,ATEMP,A,IN,ANU
D      CALL GETAS(A,ANU,MAPBMX)
```

```
C                                         PSI(0,0) SAME AS X(0,0)
        NNEXT=4
40      IJXPSI=3
50      DO 500 MAPB=1,MAPBMX
C                                         SET UP PSI(MA,0)
        MA=MAPB
        MB=0
65      N1=IN(MA+1,1)
D 70    CALL TRANS(N1,AICT1)
D 120   CALL KEEP(NNEXT,AICT1)
        IF(MA-1)9999,130,140
130     IN(MA+2,1)=NNEXT
        GO TO 500
140      IN(MA,2)=NNEXT
C                                         SET UP PSI(MA,MB)
200     MBMAX=MAPB/2
        IF(MBMAX)9999,500,210
210     DO 460 MB=1,MBMAX
        MAP1=MA
        MA=MA-1
220     N1=IN(MAP1,MB+1)
D 225   CALL TRANS(N1,AICT2)
        DO 300 I=1,MB
        MAMI=MA-I
        MBMI=MB-I
        MAMIP1=MAMI+1
        MBMIP1=MBMI+1
        J=MAMI+MBMI
         IF(J)9999,310,235
D 235   AMULT=A(I+1,J)
250     NG=IN(MAMIP1,MBMIP1)
D       CALL TRANS(NG,AICT1)
D       IF(AMULT-1.) 255,300,255
D 255   CALL MPCONS(AMULT,AICT1)
D 300   CALL ADD(AICT1,AICT2)
D 310   CALL KEEP(NNEXT,AICT2)
        IF(MB-MBMAX)400,450,450
400     IN(MA,MB+2)=NNEXT
        GO TO 460
450     IN(MAPB+2,1)=NNEXT
460     CONTINUE
500     CONTINUE
        NMAX=NNEXT-1
        DO 600 N=4,NMAX
D 600   AICM(N)=ATEMP(N)
9999    PRINT 9998
9998    FORMAT(15H  END  OF  PSIS)
        CALL CLOCK(2)
        RETURN
        END


CBIGPSI
C        GENERATES BIG PSIS
        SUBROUTINE BIGPSI(AICM,MAPBMX,ANU)
D       DIMENSION AICM(3750),APASC(15,15),ATEMP(3750),A(30,15)
        DIMENSION IPRINT(4),IN(15,15)
D       DIMENSION AICT1(16,16),AICT2(16,16)
        COMMON AICM,MA,MB,IJXPSI,IPRINT,APASC,ATEMP,A,IN,ANU
D       CALL GETPSI(AICM,MAPBMX,ANU)
C                                         BIGPSI(0,0) SAME AS PSI(0,0)
        NNEXT=4
40      IJXPSI=4
50      DO 500 MAPB=1,MAPBMX
        MBMAX=MAPB/2
        DO 400 MB=0,MBMAX
        MA=MAPB-MB
        MAMB=MA-MB
        N1=IN(MAMB+1,1)
D       CALL TRANS(N1,AICT2)
        IF(MB) 9900,350,100
100     DO300L=1,MB
D       BMLP1=MB-L+1
D       COEFF=ANU/BMLP1
D       BML=BMLP1-1.
D       CALL MPCONS(COEFF,AICT2)
D       CALL MPMN(-2.,AICT2,AICT1)
D       CALL MPCONS(BML,AICT2)
D       CALL ADD(AICT1,AICT2)
        MAMBPL=MAMB+L
        N2=IN(MAMBPL+1,L+1)
D       CALL TRANS(N2,AICT1)
D       CALL ADD(AICT1,AICT2)
300     CONTINUE
        NNEXT = IN(MA+1,MB+1)
D 350   CALL KEEP(NNEXT,AICT2)
400     CONTINUE
500     CONTINUE
        DO600 N=1,NNEXT
D 600   AICM(N)=ATEMP(N)
9900    PRINT9910
9910    FORMAT(15H END OF BIGKSIS)
        CALL CLOCK(2)
        RETURN
        END
```

```
CMAIN2
C THIS MAIN PROGRAM PUTS X-COEFFS ON TAPE B6
D     DIMENSION AICM(3750),APASC(15,15),ATEMP(3750),A(30,15)
      DIMENSION IPRINT(4),IN(15,15)
      COMMON AICM,MA,MB,IJXPSI,IPRINT,APASC,ATEMP,A,IN,ANU
      REWIND 11
    1 READ 10,MAPBMX,ANU,J,(IPRINT(I),I=1,4)
   10 FORMAT(I2,F15.8,5I4)
      PRINT 11,ANU,MAPBMX
   11 FORMAT(15H VALUE FOR NU=  F15.8,/,
     2 40H THE MAXIMUM VALUE OF ALPHA PLUS BETA IS I4)
      IF(J)20,15,20
   15 REWIND 12
      WRITE TAPE 12,MAPBMX
      WRITE TAPE 11,MAPBMX,ANU,(IPRINT(I),I=1,4)
D     CALL GETXS(AICM,MAPBMX)
      PRINT 16,MAPBMX
   16 FORMAT(/,29H THE X-COEFFICIENTS TO ORDER I2,
     2 41H HAVE BEEN COMPUTED AND STORED ON TAPE B6)
      REWIND 12
      READ TAPE 12,MAPBTP
      GOTO 30
   20 REWIND 12
      READ TAPE 12,MAPBTP
      IF(MAPBTP-MAPBMX)25,27,27
   25 PRINT 26
   26 FORMAT(104H THE ORDER OF POLYNOMIAL REQUIRED EXCEEDS THE ORDER OF
     2THE X-COEFFIENTS AVAILABLE.THIS WILL BE REMEDIED   )
      GOTO 15
   27 IPRINT(1)=0
      IPRINT(2)=0
      WRITE TAPE 11,MAPBMX,ANU,(IPRINT(I),I=1,4)
   30 NH=0
      DO 40 MAPB=0,MAPBMX
      NH=(MAPB+3)*(MAPB+2)/2*(MAPB/2+1)+NH
   40 CONTINUE
D     AICM(1)=1.0
D     AICM(2)=1.0
D     AICM(3)=1.0
      IN(1,1)=1
      DO 50 N=4,NH
      READ TAPE 12,COEFF
D     AICM(N)=COEFF
   50 CONTINUE
D     CALL BIGPSI(AICM,MAPBMX,ANU)
      GOTO 1
      END
```

References

ANDOYER, H.
1903. Contribution à la théorie des petites planètes dont le moyen mouvement est sensiblement double de celui de Jupiter. Bull. Astron., Vol. XX, pp. 321-356.

ANDOYER, H.
1923. Cours de Mecanique Céleste. Vol. 1, Gauthier-Villars et C[ie], Paris.

BARNETT, M. P., MOSS, D. J., LUCE, D. A., and KELLEY, K. L.
1963. Computer controlled printing. Proceedings Spring Joint Computer Conference, AFIPS, Vol. 23, pp. 263-288.

BOGOLIUBOV, N. N., and MITROPOLSKY, Y. A.
1958. Asymptotic Methods in the Theory of Non-linear Oscillations. Eng. trans., Hindustan Publ. Corp., Delhi, 1961.

BOQUET, F.
1889. Developpement de la fonction perturbatrice, calcul des termes du huitième ordre. Annales Obs. Paris, Mémoires. Vol. XIX, pp. B.1-B.75.

BOYS, S. F., COOK, G. B., REEVES, C. M., and SHAVITT, I.
1956. Automatic fundamental calculations of molecular structure. Nature, Vol. 178, pp. 1207-1209.

DELAUNAY, Ch. E.
1860. Théorie du Mouvement de la Lune. I. Mémoires Acad. Sci. Inst. Imp. France, Vol. XXVIII.

DELAUNAY, Ch. E.
1867. Théorie du Mouvement de la Lune. II. Mémoires Acad. Sci. Inst. Imp. France, Vol. XXIX.

HAGIHARA, Y.
1961. Gaps in the distribution of asteroids. Smithsonian Contr. Astrophys., Vol. 5, No. 6, pp. 59-67.

IZSAK, I. G., and BENIMA, B.
1963. Laplace coefficients and their Newcomb derivatives. Smithsonian Astrophys. Obs. Special Report No. 129.

LE VERRIER, U. J.
1855. Développement de la fonction qui sert de Base au calcul des perturbations des mouvement des planètes. Annales Obs. Imperial Paris. Vol. I, Mallet-Bachelier, Paris.

NEWCOMB, S.
1895. A development of the perturbative function in cosines of multiples of the mean anomalies and of angles between the perihelia and common node and in powers of the eccentricities and mutual inclination. Astron. Papers of the Amer. Ephemeris and Nautical Almanac. Vol. V, pp. 1-48.

PLUMMER, H. C.
1918. An Introductory Treatise on Dynamical Astronomy. Cambridge University Press.

POINCARÉ, H.
1893. Les Méthodes Nouvelles de la Mécanique Céleste. Gauthier-Villars et Fils, Paris. Vol. II.

POINCARÉ, H.
1902. Sur les planètes du type d'Hecube. Bull. Astron., Vol. XIX, pp. 289-310.

POINCARE, H.
1907. Leçons de Mécanique Céleste. Gauthier-Villars, Paris, Vol. II.

SHARAF, S. G.
1955. Theory of the motion of Pluto. First part. Trans. Inst. Theor. Astron. Leningrad, No. 4, pp. 3-131.

TISSERAND, F.
1880. Mémoire sur le développement de la fonction perturbatrice dans le cas ou l'inclinasion mutuelle des orbites est considerable. Application aux perturbations produites sur Pallas par Jupiter. Annales Obs. Paris, Mémoires. Vol. XV, pp. C.1-C.52.

ZEIPEL, H. v.
1912. Sur le calcul des opérateurs de Newcomb. Ark. Mat. Astro. Fys., Vol. 8, No. 19, 9 pp.

ZEIPEL, H. v.
1916. Recherches sur le mouvement des petites planètes. Ark. Mat. Astro. Fys., Vol. 11, No. 1, 58 pp.

ZEIPEL, H. v.
1923. Entwicklung der Störungsfunktion. Encyklopädie der mathematischen Wissenschaften. Sechster Band: Geodäsie, Geophysik und Astronomie. Zweiter Teil: Astronomie. Redigiert von K. Schwarzschild und S. Oppenheim (in Wien), Leipzig, Verlag und Druck von B. G. Teubner, pp. 557-665.

4.1 Newcomb operators with integer coefficients

$$J^{n,k}_{0,0} = 1$$

$$J^{n,k}_{1,0} = 2k-n$$

$$J^{n,k}_{2,0} = 5k+4k^2+(-3-4k)n+n^2$$

$$J^{n,k}_{1,1} = -4k^2+n+n^2$$

$$J^{n,k}_{3,0} = 26k+30k^2+8k^3+(-17-33k-12k^2)n+(9+6k)n^2-n^3$$

$$J^{n,k}_{2,1} = -2k-10k^2-8k^3+(3+5k+4k^2)n+(1+2k)n^2-n^3$$

$$J^{n,k}_{4,0} = 206k+283k^2+120k^3+16k^4+(-142-330k-192k^2-32k^3)n+(95+102k+24k^2)n^2 +(-18-8k)n^3+n^4$$

$$J^{n,k}_{3,1} = -22k-64k^2-60k^3-16k^4+(22+47k+48k^2+16k^3)n+(-1+3k)n^2+(-6-4k)n^3+n^4$$

$$J^{n,k}_{2,2} = -9k^2+16k^4+2n+(-1-8k^2)n^2-2n^3+n^4$$

$$J^{n,k}_{5,0} = 2{,}194k+3{,}360k^2+1{,}790k^3+400k^4+32k^5+(-1{,}569-4{,}080k-2{,}995k^2-840k^3-80k^4)n +(1{,}220+1{,}660k+660k^2+80k^3)n^2+(-305-230k-40k^2)n^3+(30+10k)n^4-n^5$$

$$J^{n,k}_{4,1} = -258k-648k^2-614k^3-240k^4-32k^5+(231+572k+617k^2+296k^3+48k^4)n +(-68-76k-60k^2-16k^3)n^2+(-41-42k-8k^2)n^3+(14+6k)n^4-n^5$$

$$J^{n,k}_{3,2} = 10k-12k^2+26k^3+80k^4+32k^5+(3-10k-9k^2-24k^3-16k^4)n+(-8-12k-36k^2-16k^3)n^2 +(-5+2k+8k^2)n^3+(6+2k)n^4-n^5$$

$$J^{n,k}_{6,0} = 29{,}352k+48{,}538k^2+29{,}835k^3+8{,}660k^4+1{,}200k^5+64k^6 +(-21{,}576-60{,}752k-51{,}615k^2-18{,}860k^3-3{,}120k^4-192k^5)n +(18{,}694+29{,}535k+15{,}345k^2+3{,}240k^3+240k^4)n^2 +(-5{,}595-5{,}530k-1{,}680k^2-160k^3)n^3+(745+435k+60k^2)n^4+(-45-12k)n^5+n^6$$

$$J^{n,k}_{5,1} = -3{,}608k-8{,}588k^2-8{,}200k^3-3{,}740k^4-800k^5-64k^6 +(3{,}096+8{,}454k+9{,}535k^2+5{,}280k^3+1{,}360k^4+128k^5)n +(-1{,}466-2{,}280k-1{,}765k^2-640k^3-80k^4)n^2+(-255-340k-80k^2)n^3 +(185+130k+20k^2)n^4+(-25-8k)n^5+n^6$$

$$J^{n,k}_{4,2} = 136k+314k^2+593k^3+788k^4+400k^5+64k^6+(-72-344k-399k^2-452k^3-304k^4-64k^5)n +(-58-115k-255k^2-136k^3-16k^4)n^2+(5+66k+112k^2+32k^3)n^3+(41+17k-4k^2)n^4 +(-13-4k)n^5+n^6$$

$$J^{n,k}_{3,3} = -172k^2+196k^4-64k^6+(24+39k^2-48k^4)n+(-26-93k^2+48k^4)n^2+(-15+48k^2)n^3 +(25-12k^2)n^4-9n^5+n^6$$

$$J^{n,k}_{7,0} = 472{,}730k+828{,}758k^2+563{,}486k^3+193{,}130k^4+35{,}560k^5+3{,}360k^6+128k^7 +(-355{,}081-1{,}062{,}285k-1{,}000{,}727k^2-432{,}775k^3-95{,}340k^4-10{,}416k^5-448k^6)n +(334{,}369+587{,}230k+361{,}935k^2+101{,}990k^3+13{,}440k^4+672k^5)n^2 +(-113{,}974-133{,}945k-54{,}425k^2-9{,}240k^3-560k^4)n^3 +(18{,}515+14{,}490k+3{,}570k^2+280k^3)n^4+(-1{,}540-735k-84k^2)n^5+(63+14k)n^6-n^7$$

$$J^{n,k}_{6,1} = -59{,}570k-138{,}970k^2-134{,}246k^3-66{,}790k^4-17{,}800k^5-2{,}400k^6-128k^7$$
$$+(50{,}195+147{,}149k+172{,}821k^2+104{,}095k^3+33{,}140k^4+5{,}232k^5+320k^6)n$$
$$+(-30{,}679-55{,}026k-45{,}045k^2-19{,}030k^3-3{,}840k^4-288k^5)n^2$$
$$+(326-175k+995k^2+600k^3+80k^4)n^3+(2{,}515+2{,}270k+570k^2+40k^3)n^4$$
$$+(-520-285k-36k^2)n^5+(39+10k)n^6-n^7$$

$$J^{n,k}_{5,2} = 2{,}850k+8{,}262k^2+12{,}246k^3+11{,}610k^4+5{,}960k^5+1{,}440k^6+128k^7$$
$$+(-2{,}229-8{,}045k-10{,}755k^2-9{,}695k^3-5{,}900k^4-1{,}776k^5-192k^6)n$$
$$+(-55-182k-1{,}485k^2-810k^3+32k^5)n^2+(446+1{,}295k+1{,}675k^2+680k^3+80k^4)n^3$$
$$+(275+50k-150k^2-40k^3)n^4+(-160-75k-4k^2)n^5+(23+6k)n^6-n^7$$

$$J^{n,k}_{4,3} = 214k-842k^2-494k^3+874k^4-40k^5-480k^6-128k^7$$
$$+(127+109k+481k^2-89k^3-364k^4+48k^5+64k^6)n$$
$$+(-159+26k-441k^2+90k^3+384k^4+96k^5)n^2+(-46+57k+351k^2+24k^3-48k^4)n^3$$
$$+(147-10k-126k^2-24k^3)n^4+(-76-9k+12k^2)n^5+(15+2k)n^6-n^7$$

$$J^{n,k}_{8,0} = 8{,}902{,}448k+16{,}352{,}684k^2+12{,}000{,}604k^3+4{,}628{,}057k^4+1{,}023{,}120k^5+130{,}592k^6$$
$$+8{,}960k^7+256k^8$$
$$+(-6{,}805{,}296-21{,}370{,}136k-21{,}771{,}428k^2-10{,}613{,}820k^3-2{,}812{,}320k^4$$
$$-415{,}744k^5-32{,}256k^6-1{,}024k^7)n$$
$$+(6{,}852{,}460+13{,}040{,}916k+9{,}077{,}782k^2+3{,}082{,}100k^3+550{,}480k^4+49{,}728k^5$$
$$+1{,}792k^6)n^2$$
$$+(-2{,}581{,}964-3{,}433{,}388k-1{,}683{,}780k^2-388{,}080k^3-42{,}560k^4-1{,}792k^5)n^3$$
$$+(484{,}729+458{,}640k+153{,}650k^2+21{,}840k^3+1{,}120k^4)n^4$$
$$+(-49{,}840-32{,}396k-6{,}720k^2-448k^3)n^5+(2{,}842+1{,}148k+112k^2)n^6+(-84-16k)n^7$$
$$+n^8$$

$$J^{n,k}_{7,1} = -1{,}139{,}472k-2{,}643{,}828k^2-2{,}590{,}518k^3-1{,}366{,}792k^4-415{,}380k^5-72{,}464k^6$$
$$-6{,}720k^7-256k^8$$
$$+(952{,}848+2{,}950{,}800k+3{,}594{,}304k^2+2{,}298{,}653k^3+830{,}480k^4+168{,}896k^5$$
$$+17{,}920k^6+768k^7)n$$
$$+(-687{,}996-1{,}351{,}154k-1{,}171{,}023k^2-546{,}875k^3-138{,}880k^4-17{,}808k^5$$
$$-896k^6)n^2$$
$$+(84{,}280+114{,}793k+89{,}250k^2+36{,}260k^3+6{,}720k^4+448k^5)n^3$$
$$+(33{,}649+35{,}525k+10{,}745k^2+980k^3)n^4+(-10{,}640-7{,}693k-1{,}680k^2-112k^3)n^5$$
$$+(1{,}162+525k+56k^2)n^6+(-56-12k)n^7+n^8$$

$$J^{n,k}_{6,2} = 65{,}968k+192{,}608k^2+262{,}622k^3+218{,}177k^4+109{,}320k^5+30{,}944k^6+4{,}480k^7+256k^8$$
$$+(-55{,}728-191{,}916k-269{,}444k^2-230{,}926k^3-131{,}680k^4-44{,}544k^5-7{,}680k^6$$
$$-512k^7)n$$
$$+(15{,}052+31{,}906k+18{,}306k^2+11{,}530k^3+8{,}400k^4+2{,}592k^5+256k^6)n^2$$
$$+(8{,}612+22{,}074k+26{,}580k^2+12{,}360k^3+2{,}240k^4+128k^5)n^3$$
$$+(889-2{,}560k-4{,}290k^2-1{,}560k^3-160k^4)n^4+(-2{,}000-1{,}062k+32k^3)n^5$$
$$+(442+190k+16k^2)n^6+(-36-8k)n^7+n^8$$

$$J^{n,k}_{5,3} = 688k-6{,}164k^2-9{,}586k^3-1{,}352k^4-4{,}380k^5-6{,}032k^6-2{,}240k^7-256k^8$$
$$+(1{,}296+2{,}032k+8{,}768k^2+3{,}919k^3-880k^4+1{,}856k^5+1{,}536k^6+256k^7)n$$
$$+(-1{,}084+642k-1{,}671k^2+2{,}695k^3+4{,}480k^4+1{,}488k^5+128k^6)n^2$$
$$+(-168+275k+2{,}610k^2-180k^3-960k^4-192k^5)n^3+(1{,}009-385k-1{,}535k^2-420k^3)n^4$$
$$+(-720-79k+240k^2+48k^3)n^5+(202+47k-8k^2)n^6+(-24-4k)n^7+n^8$$

$$J^{n,k}_{4,4} = -7{,}140k^2+7{,}801k^4-2{,}272k^6+256k^8+(720+2{,}012k^2-2{,}720k^4+512k^6)n$$
$$+(-1{,}044-3{,}674k^2+2{,}128k^4-256k^6)n^2+(-140+2{,}268k^2-576k^4)n^3$$
$$+(889-958k^2+96k^4)n^4+(-560+192k^2)n^5+(154-16k^2)n^6-20n^7+n^8$$

4.2 Newcomb operators with rational coefficients

$$X^{n,k}_{0,0} = 0.09999999\mathrm{x}10$$

$$X^{n,k}_{1,0} = 0.09999999\mathrm{x}10k-0.50000000n$$

$$X^{n,k}_{2,0} = 0.62500000k+0.50000000k^2+(-0.37500000-0.50000000k)n+0.12500000n^2$$

$$X^{n,k}_{1,1} = -0.09999999\mathrm{x}10k^2+0.25000000n+0.25000000n^2$$

$$\begin{aligned} X^{n,k}_{3,0} = {} & 0.54166666k+0.62500000k^2+0.16666666k^3 \\ & +(-0.35416666-0.68750000k-0.25000000k^2)n+(0.18750000+0.12500000k)n^2 \\ & -0.20833333\mathrm{x}10^{-1}n^3 \end{aligned}$$

$$\begin{aligned} X^{n,k}_{2,1} = {} & -0.12500000k-0.62500000k^2-0.50000000k^3 \\ & +(0.18750000+0.31250000k+0.25000000k^2)n \\ & +(0.62500000\mathrm{x}10^{-1}+0.12500000k)n^2-0.62500000\mathrm{x}10^{-1}n^3 \end{aligned}$$

$$\begin{aligned} X^{n,k}_{4,0} = {} & 0.53645833k+0.73697916k^2+0.31250000k^3+0.41666666\mathrm{x}10^{-1}k^4 \\ & +(-0.36979166-0.85937500k-0.50000000k^2-0.83333333\mathrm{x}10^{-1}k^3)n \\ & +(0.24739583+0.26562500k+0.62500000\mathrm{x}10^{-1}k^2)n^2 \\ & +(-0.46875000\mathrm{x}10^{-1}-0.20833333\mathrm{x}10^{-1}k)n^3+0.26041666\mathrm{x}10^{-2}n^4 \end{aligned}$$

$$\begin{aligned} X^{n,k}_{3,1} = {} & -0.22916666k-0.66666666k^2-0.62500000k^3-0.16666666k^4 \\ & +(0.22916666+0.48958333k+0.50000000k^2+0.16666666k^3)n \\ & +(-0.10416666\mathrm{x}10^{-1}+0.31250000\mathrm{x}10^{-1}k)n^2 \\ & +(-0.62500000\mathrm{x}10^{-1}-0.41666666\mathrm{x}10^{-1}k)n^3+0.10416666\mathrm{x}10^{-1}n^4 \end{aligned}$$

$$\begin{aligned} X^{n,k}_{2,2} = {} & -0.14062500k^2+0.25000000k^4+0.31250000\mathrm{x}10^{-1}n \\ & +(-0.15625000\mathrm{x}10^{-1}-0.12500000k^2)n^2-0.31250000\mathrm{x}10^{-1}n^3+0.15625000\mathrm{x}10^{-1}n^4 \end{aligned}$$

$$\begin{aligned} X^{n,k}_{5,0} = {} & 0.57135417k+0.87500000k^2+0.46614583k^3+0.10416666k^4+0.83333333\mathrm{x}10^{-2}k^5 \\ & +(-0.40859374-0.10625000\mathrm{x}10k-0.77994791k^2-0.21875000k^3 \\ & -0.20833333\mathrm{x}10^{-1}k^4)n \\ & +(0.31770833+0.43229166k+0.17187500k^2+0.20833333\mathrm{x}10^{-1}k^3)n^2 \\ & +(-0.79427083\mathrm{x}10^{-1}-0.59895833\mathrm{x}10^{-1}k-0.10416666\mathrm{x}10^{-1}k^2)n^3 \\ & +(0.78125000\mathrm{x}10^{-2}+0.26041666\mathrm{x}10^{-2}k)n^4-0.26041666\mathrm{x}10^{-3}n^5 \end{aligned}$$

$$\begin{aligned} X^{n,k}_{4,1} = {} & -0.33593750k-0.84375000k^2-0.79947916k^3-0.31250000k^4-0.41666666\mathrm{x}10^{-1}k^5 \\ & +(0.30078125+0.74479166k+0.80338541k^2+0.38541666k^3+0.62500000\mathrm{x}10^{-1}k^4)n \\ & +(-0.88541666\mathrm{x}10^{-1}-0.98958333\mathrm{x}10^{-1}k-0.78125000\mathrm{x}10^{-1}k^2 \\ & -0.20833333\mathrm{x}10^{-1}k^3)n^2 \\ & +(-0.53385416\mathrm{x}10^{-1}-0.54687500\mathrm{x}10^{-1}k-0.10416666\mathrm{x}10^{-1}k^2)n^3 \\ & +(0.18229166\mathrm{x}10^{-1}+0.78125000\mathrm{x}10^{-2}k)n^4-0.13020833\mathrm{x}10^{-2}n^5 \end{aligned}$$

$$\begin{aligned} X^{n,k}_{3,2} = {} & 0.26041666\mathrm{x}10^{-1}k-0.31250000\mathrm{x}10^{-1}k^2+0.67708333\mathrm{x}10^{-1}k^3+0.20833333k^4 \\ & +0.83333333\mathrm{x}10^{-1}k^5 \\ & +(0.78125000\mathrm{x}10^{-2}-0.26041666\mathrm{x}10^{-1}k-0.23437500\mathrm{x}10^{-1}k^2-0.62500000\mathrm{x}10^{-1}k^3 \\ & -0.41666666\mathrm{x}10^{-1}k^4)n \\ & +(-0.20833333\mathrm{x}10^{-1}-0.31250000\mathrm{x}10^{-1}k-0.93750000\mathrm{x}10^{-1}k^2 \\ & -0.41666666\mathrm{x}10^{-1}k^3)n^2 \\ & +(-0.13020833\mathrm{x}10^{-1}+0.52083333\mathrm{x}10^{-2}k+0.20833333\mathrm{x}10^{-1}k^2)n^3 \\ & +(0.15625000\mathrm{x}10^{-1}+0.52083333\mathrm{x}10^{-2}k)n^4-0.26041666\mathrm{x}10^{-2}n^5 \end{aligned}$$

$$
\begin{aligned}
X^{n,k}_{6,0} = {} & 0.63697916k+0.10533419\times10k^{2}+0.64746094k^{3}+0.18793403k^{4} \\
& +0.26041666\times10^{-1}k^{5}+0.13888888\times10^{-2}k^{6} \\
& +(-0.46822916-0.13184027\times10k-0.11201172\times10k^{2}-0.40928819k^{3} \\
& -0.67708333\times10^{-1}k^{4}-0.41666666\times10^{-2}k^{5})n \\
& +(0.40568576+0.64095052k+0.33300781k^{2}+0.70312500\times10^{-1}k^{3} \\
& +0.52083333\times10^{-2}k^{4})n^{2} \\
& +(-0.12141927-0.12000868k-0.36458333\times10^{-1}k^{2}-0.34722222\times10^{-2}k^{3})n^{3} \\
& +(0.16167534\times10^{-1}+0.94401041\times10^{-2}k+0.13020833\times10^{-2}k^{2})n^{4} \\
& +(-0.97656250\times10^{-3}-0.26041666\times10^{-3}k)n^{5}+0.21701388\times10^{-4}n^{6}
\end{aligned}
$$

$$
\begin{aligned}
X^{n,k}_{5,1} = {} & -0.46979167k-0.11182291\times10k^{2}-0.10677083\times10k^{3}-0.48697916k^{4}-0.10416666k^{5} \\
& -0.83333333\times10^{-2}k^{6} \\
& +(0.40312500+0.11007812\times10k+0.12415364\times10k^{2}+0.68750000k^{3}+0.17708333k^{4} \\
& +0.16666666\times10^{-1}k^{5})n \\
& +(-0.19088541-0.29687500k-0.22981770k^{2}-0.83333333\times10^{-1}k^{3} \\
& -0.10416666\times10^{-1}k^{4})n^{2} \\
& +(-0.33203125\times10^{-1}-0.44270833\times10^{-1}k-0.10416666\times10^{-1}k^{2})n^{3} \\
& +(0.24088541\times10^{-1}+0.16927083\times10^{-1}k+0.26041666\times10^{-2}k^{2})n^{4} \\
& +(-0.32552083\times10^{-2}-0.10416666\times10^{-2}k)n^{5}+0.13020833\times10^{-3}n^{6}
\end{aligned}
$$

$$
\begin{aligned}
X^{n,k}_{4,2} = {} & 0.44270833\times10^{-1}k+0.10221354k^{2}+0.19303385k^{3}+0.25651041k^{4}+0.13020833k^{5} \\
& +0.20833333\times10^{-1}k^{6} \\
& +(-0.23437500\times10^{-1}-0.11197916k-0.12988281k^{2}-0.14713541k^{3} \\
& -0.98958333\times10^{-1}k^{4}-0.20833333\times10^{-1}k^{5})n \\
& +(-0.18880208\times10^{-1}-0.37434895\times10^{-1}k-0.83007813\times10^{-1}k^{2} \\
& -0.44270833\times10^{-1}k^{3}-0.52083333\times10^{-2}k^{4})n^{2} \\
& +(0.16276041\times10^{-2}+0.21484375\times10^{-1}k+0.36458333\times10^{-1}k^{2} \\
& +0.10416666\times10^{-1}k^{3})n^{3} \\
& +(0.13346354\times10^{-1}+0.55338541\times10^{-2}k-0.13020833\times10^{-2}k^{2})n^{4} \\
& +(-0.42317708\times10^{-2}-0.13020833\times10^{-2}k)n^{5}+0.32552083\times10^{-3}n^{6}
\end{aligned}
$$

$$
\begin{aligned}
X^{n,k}_{3,3} = {} & -0.74652777\times10^{-1}k^{2}+0.85069444\times10^{-1}k^{4}-0.27777778\times10^{-1}k^{6} \\
& +(0.10416666\times10^{-1}+0.16927083\times10^{-1}k^{2}-0.20833333\times10^{-1}k^{4})n \\
& +(-0.11284722\times10^{-1}-0.40364583\times10^{-1}k^{2}+0.20833333\times10^{-1}k^{4})n^{2} \\
& +(-0.65104166\times10^{-2}+0.20833333\times10^{-1}k^{2})n^{3} \\
& +(0.10850694\times10^{-1}-0.52083333\times10^{-2}k^{2})n^{4}-0.39062500\times10^{-2}n^{5} \\
& +0.43402778\times10^{-3}n^{6}
\end{aligned}
$$

$$
\begin{aligned}
X^{n,k}_{7,0} = {} & 0.73277839k+0.12846570\times10k^{2}+0.87345920k^{3}+0.29937065k^{4} \\
& +0.55121527\times10^{-1}k^{5}+0.52083333\times10^{-2}k^{6}+0.19841269\times10^{-3}k^{7} \\
& +(-0.55041077-0.16466471\times10k-0.15512261\times10k^{2}-0.67084418k^{3}-0.14778645k^{4} \\
& -0.16145833\times10^{-1}k^{5}-0.69444443\times10^{-3}k^{6})n \\
& +(0.51830512+0.91026475k+0.56103516k^{2}+0.15809461k^{3}+0.20833333\times10^{-1}k^{4} \\
& +0.10416666\times10^{-2}k^{5})n^{2} \\
& +(-0.17667101-0.20762803k-0.84364148\times10^{-1}k^{2}-0.14322916\times10^{-1}k^{3} \\
& -0.86805555\times10^{-3}k^{4})n^{3} \\
& +(0.28700086\times10^{-1}+0.22460938\times10^{-1}k+0.55338541\times10^{-2}k^{2} \\
& +0.43402778\times10^{-3}k^{3})n^{4} \\
& +(-0.23871527\times10^{-2}-0.11393229\times10^{-2}k-0.13020833\times10^{-3}k^{2})n^{5} \\
& +(0.97656249\times10^{-4}+0.21701388\times10^{-4}k)n^{6}-0.15500992\times10^{-5}n^{7}
\end{aligned}
$$

$$
\begin{aligned}
X^{n,k}_{6,1} = {} & -0.64637586k-0.15079209x10k^{2}-0.14566623x10k^{3}-0.72471788k^{4}-0.19314235k^{5} \\
& -0.26041666x10^{-1}k^{6}-0.13888888x10^{-2}k^{7} \\
& +(0.54465061+0.15966688x10k+0.18752278x10k^{2}+0.11295030x10k^{3} \\
& +0.35959201k^{4}+0.56770833x10^{-1}k^{5}+0.34722222x10^{-2}k^{6})n \\
& +(-0.33288845-0.59707031k-0.48876953k^{2}-0.20648871k^{3}-0.41666666x10^{-1}k^{4} \\
& -0.31249999x10^{-2}k^{5})n^{2} \\
& +(0.35373264x10^{-2}-0.18988715x10^{-2}k+0.10796440x10^{-1}k^{2}+0.65104166x10^{-2}k^{3} \\
& +0.86805555x10^{-3}k^{4})n^{3} \\
& +(0.27289496x10^{-1}+0.24631076x10^{-1}k+0.61848958x10^{-2}k^{2} \\
& +0.43402778x10^{-3}k^{3})n^{4} \\
& +(-0.56423610x10^{-2}-0.30924479x10^{-2}k-0.39062499x10^{-3}k^{2})n^{5} \\
& +(0.42317708x10^{-3}+0.10850694x10^{-3}k)n^{6}-0.10850694x10^{-4}n^{7}
\end{aligned}
$$

$$
\begin{aligned}
X^{n,k}_{5,2} = {} & 0.92773438x10^{-1}k+0.26894531k^{2}+0.39863281k^{3}+0.37792969k^{4}+0.19401041k^{5} \\
& +0.46875000x10^{-1}k^{6}+0.41666666x10^{-2}k^{7} \\
& +(-0.72558593x10^{-1}-0.26188151k-0.35009766k^{2}-0.31559245k^{3}-0.19205729k^{4} \\
& -0.57812499x10^{-1}k^{5}-0.62499999x10^{-2}k^{6})n \\
& +(-0.17903645x10^{-2}-0.59244791x10^{-2}k-0.48339844x10^{-1}k^{2} \\
& -0.26367188x10^{-1}k^{3}+0.10416666x10^{-2}k^{5})n^{2} \\
& +(0.14518229x10^{-1}+0.42154948x10^{-1}k+0.54524739x10^{-1}k^{2}+0.22135416x10^{-1}k^{3} \\
& +0.26041666x10^{-2}k^{4})n^{3} \\
& +(0.89518228x10^{-2}+0.16276041x10^{-2}k-0.48828125x10^{-2}k^{2} \\
& -0.13020833x10^{-2}k^{3})n^{4} \\
& +(-0.52083333x10^{-2}-0.24414063x10^{-2}k-0.13020833x10^{-3}k^{2})n^{5} \\
& +(0.74869791x10^{-3}+0.19531249x10^{-3}k)n^{6}-0.32552083x10^{-4}n^{7}
\end{aligned}
$$

$$
\begin{aligned}
X^{n,k}_{4,3} = {} & 0.11610243x10^{-1}k-0.45681423x10^{-1}k^{2}-0.26801215x10^{-1}k^{3}+0.47417534x10^{-1}k^{4} \\
& -0.21701388x10^{-2}k^{5}-0.26041666x10^{-1}k^{6}-0.69444444x10^{-2}k^{7} \\
& +(0.68901909x10^{-2}+0.59136284x10^{-2}k+0.26095919x10^{-1}k^{2}-0.48285589x10^{-2}k^{3} \\
& -0.19748263x10^{-1}k^{4}+0.26041666x10^{-2}k^{5}+0.34722222x10^{-2}k^{6})n \\
& +(-0.86263020x10^{-2}+0.14105903x10^{-2}k-0.23925781x10^{-1}k^{2} \\
& +0.48828125x10^{-2}k^{3}+0.20833333x10^{-1}k^{4}+0.52083333x10^{-2}k^{5})n^{2} \\
& +(-0.24956597x10^{-2}+0.30924479x10^{-2}k+0.19042969x10^{-1}k^{2} \\
& +0.13020833x10^{-2}k^{3}-0.26041666x10^{-2}k^{4})n^{3} \\
& +(0.79752603x10^{-2}-0.54253472x10^{-3}k-0.68359375x10^{-2}k^{2} \\
& -0.13020833x10^{-2}k^{3})n^{4} \\
& +(-0.41232638x10^{-2}-0.48828125x10^{-3}k+0.65104166x10^{-3}k^{2})n^{5} \\
& +(0.81380208x10^{-3}+0.10850694x10^{-3}k)n^{6}-0.54253472x10^{-4}n^{7}
\end{aligned}
$$

$$
\begin{aligned}
X^{n,k}_{8,0} = {} & 0.86247984k+0.15842676x10k^{2}+0.11626329x10k^{3}+0.44837172k^{4} \\
& +0.99121094x10^{-1}k^{5}+0.12651909x10^{-1}k^{6}+0.86805555x10^{-3}k^{7} \\
& +0.24801587x10^{-4}k^{8} \\
& +(-0.65930524-0.20703644x10k-0.21092420x10k^{2}-0.10282796x10k^{3} \\
& -0.27246094k^{4}-0.40277778x10^{-1}k^{5}-0.31249999x10^{-2}k^{6}-0.99206349x10^{-4}k^{7})n \\
& +(0.66387454+0.12634195x10k+0.87946641k^{2}+0.29859754k^{3} \\
& +0.53331163x10^{-1}k^{4}+0.48177082x10^{-2}k^{5}+0.17361110x10^{-3}k^{6})n^{2} \\
& +(-0.25014377-0.33263075k-0.16312663k^{2}-0.37597656x10^{-1}k^{3} \\
& -0.41232638x10^{-2}k^{4}-0.17361110x10^{-3}k^{5})n^{3} \\
& +(0.46961127x10^{-1}+0.44433594x10^{-1}k+0.14885796x10^{-1}k^{2}+0.21158854x10^{-2}k^{3} \\
& +0.10850694x10^{-3}k^{4})n^{4} \\
& +(-0.48285589x10^{-2}-0.31385633x10^{-2}k-0.65104166x10^{-3}k^{2} \\
& -0.43402777x10^{-4}k^{3})n^{5} \\
& +(0.27533636x10^{-3}+0.11121961x10^{-3}k+0.10850694x10^{-4}k^{2})n^{6} \\
& +(-0.81380208x10^{-5}-0.15500992x10^{-5}k)n^{7}+0.96881200x10^{-7}n^{8}
\end{aligned}
$$

$$
\begin{aligned}
X^{n,k}_{7,1} = {} & -0.88314731k-0.20490978\times10k^{2}-0.20077799\times10k^{3}-0.10593315\times10k^{4} \\
& -0.32194010k^{5}-0.56163194\times10^{-1}k^{6}-0.52083333\times10^{-2}k^{7}-0.19841269\times10^{-3}k^{8} \\
& +(0.73850446+0.22870163\times10k+0.27857638\times10k^{2}+0.17815700\times10k^{3} \\
& +0.64366319k^{4}+0.13090277k^{5}+0.13888889\times10^{-1}k^{6}+0.59523808\times10^{-3}k^{7})n \\
& +(-0.53323102-0.10472113\times10k-0.90760091k^{2}-0.42385525k^{3}-0.10763889k^{4} \\
& -0.13802083\times10^{-1}k^{5}-0.69444443\times10^{-3}k^{6})n^{2} \\
& +(0.65321179\times10^{-1}+0.88970269\times10^{-1}k+0.69173177\times10^{-1}k^{2}+0.28103298\times10^{-1}k^{3} \\
& +0.52083333\times10^{-2}k^{4}+0.34722222\times10^{-3}k^{5})n^{3} \\
& +(0.26079644\times10^{-1}+0.27533637\times10^{-1}k+0.83279078\times10^{-2}k^{2} \\
& +0.75954860\times10^{-3}k^{3})n^{4} \\
& +(-0.82465277\times10^{-2}-0.59624565\times10^{-2}k-0.13020833\times10^{-2}k^{2} \\
& -0.86805554\times10^{-4}k^{3})n^{5} \\
& +(0.90060763\times10^{-3}+0.40690104\times10^{-3}k+0.43402777\times10^{-4}k^{2})n^{6} \\
& +(-0.43402777\times10^{-4}-0.93005951\times10^{-5}k)n^{7}+0.77504960\times10^{-6}n^{8}
\end{aligned}
$$

$$
\begin{aligned}
X^{n,k}_{6,2} = {} & 0.17894965k+0.52248263k^{2}+0.71240777k^{3}+0.59184299k^{4}+0.29654948k^{5} \\
& +0.83940972\times10^{-1}k^{6}+0.12152778\times10^{-1}k^{7}+0.69444443\times10^{-3}k^{8} \\
& +(-0.15117187-0.52060547k-0.73091362k^{2}-0.62642686k^{3}-0.35720485k^{4} \\
& -0.12083333k^{5}-0.20833333\times10^{-1}k^{6}-0.13888888\times10^{-2}k^{7})n \\
& +(0.40831163\times10^{-1}+0.86550564\times10^{-1}k+0.49658202\times10^{-1}k^{2}+0.31277126\times10^{-1}k^{3} \\
& +0.22786458\times10^{-1}k^{4}+0.70312499\times10^{-2}k^{5}+0.69444443\times10^{-3}k^{6})n^{2} \\
& +(0.23361545\times10^{-1}+0.59879557\times10^{-1}k+0.72102864\times10^{-1}k^{2}+0.33528645\times10^{-1}k^{3} \\
& +0.60763888\times10^{-2}k^{4}+0.34722222\times10^{-3}k^{5})n^{3} \\
& +(0.24115667\times10^{-2}-0.69444444\times10^{-2}k-0.11637370\times10^{-1}k^{2}-0.42317708\times10^{-2}k^{3} \\
& -0.43402778\times10^{-3}k^{4})n^{4} \\
& +(-0.54253472\times10^{-2}-0.28808593\times10^{-2}k+0.86805554\times10^{-4}k^{3})n^{5} \\
& +(0.11990017\times10^{-2}+0.51540798\times10^{-3}k+0.43402777\times10^{-4}k^{2})n^{6} \\
& +(-0.97656249\times10^{-4}-0.21701388\times10^{-4}k)n^{7}+0.27126735\times10^{-5}n^{8}
\end{aligned}
$$

$$
\begin{aligned}
X^{n,k}_{5,3} = {} & 0.37326388\times10^{-2}k-0.33441840\times10^{-1}k^{2}-0.52007378\times10^{-1}k^{3}-0.73350694\times10^{-2}k^{4} \\
& -0.23763020\times10^{-1}k^{5}-0.32725694\times10^{-1}k^{6}-0.12152778\times10^{-1}k^{7} \\
& -0.13888888\times10^{-2}k^{8} \\
& +(0.70312499\times10^{-2}+0.11024305\times10^{-1}k+0.47569444\times10^{-1}k^{2}+0.21261935\times10^{-1}k^{3} \\
& -0.47743054\times10^{-2}k^{4}+0.10069444\times10^{-1}k^{5}+0.83333333\times10^{-2}k^{6} \\
& +0.13888888\times10^{-2}k^{7})n \\
& +(-0.58810764\times10^{-2}+0.34830729\times10^{-2}k-0.90657552\times10^{-2}k^{2} \\
& +0.14621311\times10^{-1}k^{3}+0.24305555\times10^{-1}k^{4}+0.80729166\times10^{-2}k^{5} \\
& +0.69444443\times10^{-3}k^{6})n^{2} \\
& +(-0.91145833\times10^{-3}+0.14919704\times10^{-2}k+0.14160156\times10^{-1}k^{2} \\
& -0.97656250\times10^{-3}k^{3}-0.52083333\times10^{-2}k^{4}-0.10416666\times10^{-2}k^{5})n^{3} \\
& +(0.54741753\times10^{-2}-0.20887586\times10^{-2}k-0.83279078\times10^{-2}k^{2} \\
& -0.22786458\times10^{-2}k^{3})n^{4} \\
& +(-0.39062500\times10^{-2}-0.42860243\times10^{-3}k+0.13020833\times10^{-2}k^{2} \\
& +0.26041666\times10^{-3}k^{3})n^{5} \\
& +(0.10959201\times10^{-2}+0.25499132\times10^{-3}k-0.43402777\times10^{-4}k^{2})n^{6} \\
& +(-0.13020833\times10^{-3}-0.21701388\times10^{-4}k)n^{7}+0.54253472\times10^{-5}n^{8}
\end{aligned}
$$

$$
\begin{aligned}
X_{4,4}^{n,k} = {} & -0.48421223\times10^{-1}k^{2}+0.52903917\times10^{-1}k^{4}-0.15407985\times10^{-1}k^{6} \\
& +0.17361110\times10^{-2}k^{8} \\
& +(0.48828125\times10^{-2}+0.13644748\times10^{-1}k^{2}-0.18446180\times10^{-1}k^{4} \\
& +0.34722222\times10^{-2}k^{6})n \\
& +(-0.70800781\times10^{-2}-0.24915907\times10^{-1}k^{2}+0.14431423\times10^{-1}k^{4} \\
& -0.17361110\times10^{-2}k^{6})n^{2} \\
& +(-0.94943576\times10^{-3}+0.15380859\times10^{-1}k^{2}-0.39062500\times10^{-2}k^{4})n^{3} \\
& +(0.60289171\times10^{-2}-0.64968532\times10^{-2}k^{2}+0.65104166\times10^{-3}k^{4})n^{4} \\
& +(-0.37977430\times10^{-2}+0.13020833\times10^{-2}k^{2})n^{5} \\
& +(0.10443793\times10^{-2}-0.10850694\times10^{-3}k^{2})n^{6}-0.13563368\times10^{-3}n^{7} \\
& +0.67816840\times10^{-5}n^{8}
\end{aligned}
$$

$$
\begin{aligned}
X_{9,0}^{n,k} = {} & 0.10329466\times10k+0.19720044\times10k^{2}+0.15375723\times10k^{3}+0.64850803k^{4} \\
& +0.16339269k^{5}+0.25358073\times10^{-1}k^{6}+0.23799189\times10^{-2}k^{7}+0.12400793\times10^{-3}k^{8} \\
& +0.27557319\times10^{-5}k^{9} \\
& +(-0.80125679-0.26190402\times10k-0.28394803\times10k^{2}-0.15162732\times10k^{3} \\
& -0.45855453k^{4}-0.82535807\times10^{-1}k^{5}-0.87709779\times10^{-2}k^{6}-0.50843254\times10^{-3}k^{7} \\
& -0.12400793\times10^{-4}k^{8})n \\
& +(0.85298239+0.17303805\times10k+0.13214056\times10k^{2}+0.51280200k^{3}+0.11166721k^{4} \\
& +0.13834635\times10^{-1}k^{5}+0.91145833\times10^{-3}k^{6}+0.24801587\times10^{-4}k^{7})n^{2} \\
& +(-0.34838222-0.50899386k-0.28571890k^{2}-0.80396863\times10^{-1}k^{3} \\
& -0.12107566\times10^{-1}k^{4}-0.93315972\times10^{-3}k^{5}-0.28935185\times10^{-4}k^{6})n^{3} \\
& +(0.73151313\times10^{-1}+0.79335077\times10^{-1}k+0.32491048\times10^{-1}k^{2}+0.63499168\times10^{-2}k^{3} \\
& +0.59678818\times10^{-3}k^{4}+0.21701388\times10^{-4}k^{5})n^{4} \\
& +(-0.87844282\times10^{-2}-0.69892035\times10^{-2}k-0.19958495\times10^{-2}k^{2} \\
& -0.24414063\times10^{-3}k^{3}-0.10850694\times10^{-4}k^{4})n^{5} \\
& +(0.62527126\times10^{-3}+0.34812644\times10^{-3}k+0.62391493\times10^{-4}k^{2} \\
& +0.36168981\times10^{-5}k^{3})n^{6} \\
& +(-0.25996455\times10^{-4}-0.91068327\times10^{-5}k-0.77504960\times10^{-6}k^{2})n^{7} \\
& +(0.58128719\times10^{-6}+0.96881200\times10^{-7}k)n^{8}-0.53822888\times10^{-8}n^{9}
\end{aligned}
$$

$$
\begin{aligned}
X_{8,1}^{n,k} = {} & -0.12025763\times10k-0.27948567\times10k^{2}-0.27808429\times10k^{3}-0.15336697\times10k^{4} \\
& -0.50973239k^{5}-0.10428602k^{6}-0.12825520\times10^{-1}k^{7}-0.86805555\times10^{-3}k^{8} \\
& -0.24801587\times10^{-4}k^{9} \\
& +(0.10031117\times10+0.32469966\times10k+0.40865211\times10k^{2}+0.27398654\times10k^{3} \\
& +0.10798641\times10k^{4}+0.25671115k^{5}+0.36035156\times10^{-1}k^{6}+0.27405754\times10^{-2}k^{7} \\
& +0.86805554\times10^{-4}k^{8})n \\
& +(-0.81687515-0.17145132\times10k-0.15636799\times10k^{2}-0.78698459k^{3}-0.23034668k^{4} \\
& -0.38661024\times10^{-1}k^{5}-0.34288194\times10^{-2}k^{6}-0.12400793\times10^{-3}k^{7})n^{2} \\
& +(0.16459253+0.25444471k+0.18950805k^{2}+0.78253851\times10^{-1}k^{3} \\
& +0.17550997\times10^{-1}k^{4}+0.19748264\times10^{-2}k^{5}+0.86805554\times10^{-4}k^{6})n^{3} \\
& +(0.17438422\times10^{-1}+0.20267062\times10^{-1}k+0.50387911\times10^{-2}k^{2}-0.48149956\times10^{-3}k^{3} \\
& -0.27126735\times10^{-3}k^{4}-0.21701388\times10^{-4}k^{5})n^{4} \\
& +(-0.10700480\times10^{-1}-0.92244466\times10^{-2}k-0.26441786\times10^{-2}k^{2} \\
& -0.29839409\times10^{-3}k^{3}-0.10850694\times10^{-4}k^{4})n^{5} \\
& +(0.15747070\times10^{-2}+0.95757377\times10^{-3}k+0.18174913\times10^{-3}k^{2} \\
& +0.10850694\times10^{-4}k^{3})n^{6} \\
& +(-0.10918511\times10^{-3}-0.42046440\times10^{-4}k-0.38752480\times10^{-5}k^{2})n^{7} \\
& +(0.36814856\times10^{-5}+0.67816839\times10^{-6}k)n^{8}-0.48440599\times10^{-7}n^{9}
\end{aligned}
$$

$$
\begin{aligned}
X^{n.k}_{7,2} = {} & 0.31955489k+0.91537621k^2+0.11951001\times10k^3+0.93519422k^4+0.46137424k^5 \\
& +0.14105903k^6+0.25520833\times10^{-1}k^7+0.24801587\times10^{-2}k^8+0.99206349\times10^{-4}k^9 \\
& +(-0.27535284-0.94954078k-0.13664984\times10k^2-0.11594916\times10k^3-0.64569363k^4 \\
& -0.22988281k^5-0.48611110\times10^{-1}k^6-0.54563491\times10^{-2}k^7-0.24801587\times10^{-3}k^8)n \\
& +(0.12547510+0.28382626k+0.26380751k^2+0.16653646k^3+0.79372829\times10^{-1}k^4 \\
& +0.23307291\times10^{-1}k^5+0.34722222\times10^{-2}k^6+0.19841269\times10^{-3}k^7)n^2 \\
& +(0.22866675\times10^{-1}+0.63186305\times10^{-1}k+0.78987630\times10^{-1}k^2+0.39930555\times10^{-1}k^3 \\
& +0.83550346\times10^{-2}k^4+0.60763888\times10^{-3}k^5)n^3 \\
& +(-0.59895833\times10^{-2}-0.20071071\times10^{-1}k-0.22216797\times10^{-1}k^2 \\
& -0.91145832\times10^{-2}k^3-0.15190972\times10^{-2}k^4-0.86805554\times10^{-4}k^5)n^4 \\
& +(-0.46617296\times10^{-2}-0.20480685\times10^{-2}k+0.83007812\times10^{-3}k^2 \\
& +0.43402778\times10^{-3}k^3+0.43402777\times10^{-4}k^4)n^5 \\
& +(0.16059028\times10^{-2}+0.86805555\times10^{-3}k+0.10850694\times10^{-3}k^2)n^6 \\
& +(-0.19259983\times10^{-3}-0.73629711\times10^{-4}k-0.62003968\times10^{-5}k^2)n^7 \\
& +(0.10075644\times10^{-4}+0.19376240\times10^{-5}k)n^8-0.19376240\times10^{-6}n^9
\end{aligned}
$$

$$
\begin{aligned}
X^{n.k}_{6,3} = {} & -0.79182941\times10^{-2}k-0.62266710\times10^{-1}k^2-0.11032262k^3-0.88644747\times10^{-1}k^4 \\
& -0.72398545\times10^{-1}k^5-0.51041666\times10^{-1}k^6-0.19444444\times10^{-1}k^7 \\
& -0.34722222\times10^{-2}k^8-0.23148148\times10^{-3}k^9 \\
& +(0.14400227\times10^{-1}+0.43996853\times10^{-1}k+0.10228226k^2+0.90085177\times10^{-1}k^3 \\
& +0.43441207\times10^{-1}k^4+0.31532117\times10^{-1}k^5+0.17332175\times10^{-1}k^6 \\
& +0.41666666\times10^{-2}k^7+0.34722222\times10^{-3}k^8)n \\
& +(-0.48990885\times10^{-2}+0.42679397\times10^{-2}k-0.24793836\times10^{-2}k^2 \\
& +0.20717592\times10^{-1}k^3+0.27289496\times10^{-1}k^4+0.96788193\times10^{-2}k^5 \\
& +0.10416666\times10^{-2}k^6)n^2 \\
& +(-0.19323278\times10^{-2}-0.46522351\times10^{-2}k+0.35337094\times10^{-2}k^2 \\
& -0.78667533\times10^{-2}k^3-0.92954282\times10^{-2}k^4-0.26909722\times10^{-2}k^5 \\
& -0.23148148\times10^{-3}k^6)n^3 \\
& +(0.36132812\times10^{-2}-0.30481409\times10^{-2}k-0.89789496\times10^{-2}k^2-0.26222511\times10^{-2}k^3 \\
& +0.21701389\times10^{-3}k^4+0.86805554\times10^{-4}k^5)n^4 \\
& +(-0.33542209\times10^{-2}+0.14377169\times10^{-3}k+0.21267360\times10^{-2}k^2 \\
& +0.65104166\times10^{-3}k^3+0.43402777\times10^{-4}k^4)n^5 \\
& +(0.12695312\times10^{-2}+0.33998842\times10^{-3}k-0.13020833\times10^{-3}k^2 \\
& -0.28935185\times10^{-4}k^3)n^6 \\
& +(-0.21430121\times10^{-3}-0.59678819\times10^{-4}k)n^7 \\
& +(0.16276041\times10^{-4}+0.27126735\times10^{-5}k)n^8-0.45211226\times10^{-6}n^9
\end{aligned}
$$

$$
\begin{aligned}
X^{n.k}_{5,4} = {} & 0.84323458\times10^{-2}k-0.30257161\times10^{-1}k^2-0.16783311\times10^{-1}k^3+0.38612195\times10^{-1}k^4 \\
& +0.79169379\times10^{-2}k^5-0.11089409\times10^{-1}k^6-0.91145833\times10^{-3}k^7 \\
& +0.17361110\times10^{-2}k^8+0.34722222\times10^{-3}k^9 \\
& +(0.35420735\times10^{-2}+0.32253689\times10^{-2}k+0.17175292\times10^{-1}k^2-0.53955077\times10^{-2}k^3 \\
& -0.19057210\times10^{-1}k^4+0.66189235\times10^{-3}k^5+0.40147569\times10^{-2}k^6 \\
& +0.17361110\times10^{-3}k^7-0.17361110\times10^{-3}k^8)n \\
& +(-0.56911892\times10^{-2}+0.54796007\times10^{-3}k-0.19053819\times10^{-1}k^2 \\
& -0.11610243\times10^{-2}k^3+0.12017144\times10^{-1}k^4+0.54253472\times10^{-3}k^5 \\
& -0.19965278\times10^{-2}k^6-0.34722222\times10^{-3}k^7)n^2 \\
& +(0.96299913\times10^{-4}+0.11555989\times10^{-2}k+0.13689507\times10^{-1}k^2-0.25770399\times10^{-3}k^3 \\
& -0.45030382\times10^{-2}k^4-0.39062499\times10^{-3}k^5+0.17361110\times10^{-3}k^6)n^3 \\
& +(0.44949001\times10^{-2}-0.37841796\times10^{-3}k-0.63340928\times10^{-2}k^2-0.33908419\times10^{-3}k^3 \\
& +0.97656250\times10^{-3}k^4+0.13020833\times10^{-3}k^5)n^4 \\
& +(-0.34308539\times10^{-2}-0.43402777\times10^{-4}k+0.16832139\times10^{-2}k^2 \\
& +0.16276041\times10^{-3}k^3-0.65104166\times10^{-4}k^4)n^5 \\
& +(0.11745876\times10^{-2}+0.81380208\times10^{-4}k-0.22243923\times10^{-3}k^2 \\
& -0.21701388\times10^{-4}k^3)n^6 \\
& +(-0.21023220\times10^{-3}-0.18988714\times10^{-4}k+0.10850694\times10^{-4}k^2)n^7 \\
& +(0.18988714\times10^{-4}+0.13563368\times10^{-5}k)n^8-0.67816839\times10^{-6}n^9
\end{aligned}
$$

$$
\begin{aligned}
X^{n,k}_{10,0} = \; & 0.12541313\times10k + 0.24738538\times10k^{2} + 0.20272344\times10k^{3} + 0.91804938k^{4} \\
& + 0.25555501k^{5} + 0.45655404\times10^{-1}k^{6} + 0.52616825\times10^{-2}k^{7} + 0.37874090\times10^{-3}k^{8} \\
& + 0.15500992\times10^{-4}k^{9} + 0.27557319\times10^{-6}k^{10} \\
& + (-0.98501691 - 0.33310774\times10k - 0.38008564\times10k^{2} - 0.21820702\times10k^{3} \\
& - 0.72998835k^{4} - 0.15142390k^{5} - 0.19781720\times10^{-1}k^{6} - 0.15857515\times10^{-2}k^{7} \\
& - 0.71304563\times10^{-4}k^{8} - 0.13778659\times10^{-5}k^{9})n \\
& + (0.10994266\times10 + 0.23506019\times10k + 0.19323537\times10k^{2} + 0.83054125k^{3} \\
& + 0.20867075k^{4} + 0.31815592\times10^{-1}k^{5} + 0.29016565\times10^{-2}k^{6} + 0.14570932\times10^{-3}k^{7} \\
& + 0.31001984\times10^{-5}k^{8})n^{2} \\
& + (-0.48009884 - 0.75606086k - 0.47061253k^{2} - 0.15296009k^{3} - 0.28379086\times10^{-1}k^{4} \\
& - 0.30309606\times10^{-2}k^{5} - 0.17361110\times10^{-3}k^{6} - 0.41335978\times10^{-5}k^{7})n^{3} \\
& + (0.11033359 + 0.13284361k + 0.62912269\times10^{-1}k^{2} + 0.15163562\times10^{-1}k^{3} \\
& + 0.19768608\times10^{-2}k^{4} + 0.13292100\times10^{-3}k^{5} + 0.36168981\times10^{-5}k^{6})n^{4} \\
& + (-0.14948159\times10^{-1} - 0.13767993\times10^{-1}k - 0.48538207\times10^{-2}k^{2} \\
& - 0.82442671\times10^{-3}k^{3} - 0.67816839\times10^{-4}k^{4} - 0.21701389\times10^{-5}k^{5})n^{5} \\
& + (0.12526505\times10^{-2} + 0.86189552\times10^{-3}k + 0.21469681\times10^{-3}k^{2} + 0.23057725\times10^{-4}k^{3} \\
& + 0.90422453\times10^{-6}k^{4})n^{6} \\
& + (-0.65499765\times10^{-4} - 0.31922355\times10^{-4}k - 0.50378224\times10^{-5}k^{2} \\
& - 0.25834987\times10^{-6}k^{3})n^{7} \\
& + (0.20748723\times10^{-5} + 0.64183795\times10^{-6}k + 0.48440599\times10^{-7}k^{2})n^{8} \\
& + (-0.36330450\times10^{-7} - 0.53822888\times10^{-8}k)n^{9} + 0.26911444\times10^{-9}n^{10}
\end{aligned}
$$

$$
\begin{aligned}
X^{n,k}_{9,1} = \; & -0.16346598\times10k - 0.38191450\times10k^{2} - 0.38597079\times10k^{3} - 0.22062662\times10k^{4} \\
& - 0.78356571k^{5} - 0.17905024k^{6} - 0.26327401\times10^{-1}k^{7} - 0.24047205\times10^{-2}k^{8} \\
& - 0.12400793\times10^{-3}k^{9} - 0.27557319\times10^{-5}k^{10} \\
& + (0.13637604\times10 + 0.45800663\times10k + 0.59358889\times10k^{2} + 0.41395622\times10k^{3} \\
& + 0.17417270\times10k^{4} + 0.46143663k^{5} + 0.77287687\times10^{-1}k^{6} + 0.79220403\times10^{-2}k^{7} \\
& + 0.45262896\times10^{-3}k^{8} + 0.11022928\times10^{-4}k^{9})n \\
& + (-0.12180641\times10 - 0.26939501\times10k - 0.25692941\times10k^{2} - 0.13728520\times10k^{3} \\
& - 0.44400651k^{4} - 0.88248697\times10^{-1}k^{5} - 0.10490812\times10^{-1}k^{6} - 0.68204364\times10^{-3}k^{7} \\
& - 0.18601190\times10^{-4}k^{8})n^{2} \\
& + (0.31935651 + 0.53387193k + 0.40992217k^{2} + 0.17838948k^{3} + 0.45581958\times10^{-1}k^{4} \\
& + 0.66984953\times10^{-2}k^{5} + 0.52083333\times10^{-3}k^{6} + 0{,}16534391\times10^{-4}k^{7})n^{3} \\
& + (-0.34294438\times10^{-2} - 0.57987919\times10^{-2}k - 0.10409545\times10^{-1}k^{2} \\
& - 0.61634204\times10^{-2}k^{3} - 0.15439633\times10^{-2}k^{4} - 0.17361110\times10^{-3}k^{5} \\
& - 0.72337963\times10^{-5}k^{6})n^{4} \\
& + (-0.12349389\times10^{-1} - 0.12006971\times10^{-1}k - 0.39967854\times10^{-2}k^{2} \\
& - 0.55248118\times10^{-3}k^{3} - 0.27126735\times10^{-4}k^{4})n^{5} \\
& + (0.24353592\times10^{-2} + 0.17942075\times10^{-2}k + 0.45991120\times10^{-3}k^{2} + 0.48828124\times10^{-4}k^{3} \\
& + 0.18084490\times10^{-5}k^{4})n^{6} \\
& + (-0.22051775\times10^{-3} - 0.11599909\times10^{-3}k - 0.19376240\times10^{-4}k^{2} \\
& - 0.10333994\times10^{-5}k^{3})n^{7} \\
& + (0.10673078\times10^{-4} + 0.35846043\times10^{-5}k + 0.29064360\times10^{-6}k^{2})n^{8} \\
& + (-0.26642329\times10^{-6} - 0.43058310\times10^{-7}k)n^{9} + 0.26911444\times10^{-8}n^{10}
\end{aligned}
$$

$$
\begin{aligned}
X^{n,k}_{8,2} = {} & 0.54066685k+0.15202875x10k^{2}+0.19349770x10k^{3}+0.14704686x10k^{4} \\
& +0.72131525k^{5}+0.23102857k^{6}+0.47469075x10^{-1}k^{7}+0.59818327x10^{-2}k^{8} \\
& +0.41852678x10^{-3}k^{9}+0.12400793x10^{-4}k^{10} \\
& +(-0.46834542-0.16398379x10k-0.24006705x10k^{2}-0.20419721x10k^{3} \\
& -0.11333856x10k^{4}-0.41699795k^{5}-0.98727755x10^{-1}k^{6}-0.14231461x10^{-1}k^{7} \\
& -0.11253720x10^{-2}k^{8}-0.37202380x10^{-4}k^{9})n \\
& +(0.27775355+0.65618891k+0.67944156k^{2}+0.44077877k^{3}+0.19920476k^{4} \\
& +0.59606933x10^{-1}k^{5}+0.10739475x10^{-1}k^{6}+0.10323660x10^{-2}k^{7} \\
& +0.40302579x10^{-4}k^{8})n^{2} \\
& +(0.34917437x10^{-2}+0.31509642x10^{-1}k+0.55644564x10^{-1}k^{2}+0.30404662x10^{-1}k^{3} \\
& +0.52761501x10^{-2}k^{4}-0.32552083x10^{-3}k^{5}-0.17361110x10^{-3}k^{6} \\
& -0.12400793x10^{-4}k^{7})n^{3} \\
& +(-0.15229046x10^{-1}-0.36425357x10^{-1}k-0.36165194x10^{-1}k^{2} \\
& -0.15951368x10^{-1}k^{3}-0.33264160x10^{-2}k^{4}-0.31738281x10^{-3}k^{5} \\
& -0.10850694x10^{-4}k^{6})n^{4} \\
& +(-0.26639302x10^{-2}+0.67003038x10^{-3}k+0.28996785x10^{-2}k^{2} \\
& +0.12810601x10^{-2}k^{3}+0.20345052x10^{-3}k^{4}+0.10850694x10^{-4}k^{5})n^{5} \\
& +(0.18770853x10^{-2}+0.11181301x10^{-2}k+0.12902153x10^{-3}k^{2}-0.17632378x10^{-4}k^{3} \\
& -0.27126735x10^{-5}k^{4})n^{6} \\
& +(-0.31077067x10^{-3}-0.15583341x10^{-3}k-0.22088913x10^{-4}k^{2} \\
& -0.77504960x10^{-6}k^{3})n^{7} \\
& +(0.23614793x10^{-4}+0.80290294x10^{-5}k+0.62972780x10^{-6}k^{2})n \\
& +(-0.85982065x10^{-6}-0.14532179x10^{-6}k)n^{9}+0.12110150x10^{-7}n^{10}
\end{aligned}
$$

$$
\begin{aligned}
X^{n,k}_{7,3} = {} & -0.33461475x10^{-1}k-0.14399801k^{2}-0.24342602k^{3}-0.22804879k^{4}-0.16070420k^{5} \\
& -0.87936740x10^{-1}k^{6}-0.31835937x10^{-1}k^{7}-0.67336309x10^{-2}k^{8} \\
& -0.74404761x10^{-3}k^{9}-0.33068783x10^{-4}k^{10} \\
& +(0.35710151x10^{-1}+0.12827432k+0.24294291k^{2}+0.24131731k^{3}+0.15108936k^{4} \\
& +0.82103588x10^{-1}k^{5}+0.35698784x10^{-1}k^{6}+0.94204695x10^{-2}k^{7} \\
& +0.12648809x10^{-2}k^{8}+0.66137566x10^{-4}k^{9})n \\
& +(-0.10648664x10^{-1}-0.11802842x10^{-1}k-0.17678235x10^{-1}k^{2} \\
& +0.90720847x10^{-2}k^{3}+0.21425374x10^{-1}k^{4}+0.77473957x10^{-2}k^{5} \\
& +0.29658564x10^{-3}k^{6}-0.22321428x10^{-3}k^{7}-0.24801587x10^{-4}k^{8})n^{2} \\
& +(-0.52983682x10^{-2}-0.16105078x10^{-1}k-0.13099048x10^{-1}k^{2} \\
& -0.19973415x10^{-1}k^{3}-0.15661169x10^{-1}k^{4}-0.50636573x10^{-2}k^{5} \\
& -0.69444443x10^{-3}k^{6}-0.33068783x10^{-4}k^{7})n^{3} \\
& +(0.29511305x10^{-2}-0.20363136x10^{-2}k-0.73852539x10^{-2}k^{2}-0.15620478x10^{-2}k^{3} \\
& +0.96752025x10^{-3}k^{4}+0.34722222x10^{-3}k^{5}+0.28935185x10^{-4}k^{6})n^{4} \\
& +(-0.26016800x10^{-2}+0.10357892x10^{-2}k+0.30802408x10^{-2}k^{2} \\
& +0.11049623x10^{-2}k^{3}+0.10850694x10^{-3}k^{4})n^{5} \\
& +(0.13163248x10^{-2}+0.29613353x10^{-3}k-0.29342086x10^{-3}k^{2}-0.97656249x10^{-4}k^{3} \\
& -0.72337963x10^{-5}k^{4})n^{6} \\
& +(-0.29522931x10^{-3}-0.10178985x10^{-3}k+0.31001984x10^{-5}k^{2} \\
& +0.20667989x10^{-5}k^{3})n^{7} \\
& +(0.31970796x10^{-4}+0.91068327x10^{-5}k+0.38752480x10^{-6}k^{2})n^{8} \\
& +(-0.16469803x10^{-5}-0.25834987x10^{-6}k)n^{9}+0.32293733x10^{-7}n^{10}
\end{aligned}
$$

$$
\begin{aligned}
X_{6,4}^{n,k} = {} & 0.48972800\times10^{-2}k-0.13612196\times10^{-1}k^{2}-0.15419514\times10^{-1}k^{3}+0.27002179\times10^{-1}k^{4} \\
& +0.13950432\times10^{-1}k^{5}-0.40913899\times10^{-2}k^{6}+0.26584201\times10^{-3}k^{7} \\
& +0.21050347\times10^{-2}k^{8}+0.65104166\times10^{-3}k^{9}+0.57870370\times10^{-4}k^{10} \\
& +(0.25245949\times10^{-2}-0.87438512\times10^{-3}k+0.17267749\times10^{-1}k^{2}-0.46160662\times10^{-2}k^{3} \\
& -0.21811195\times10^{-1}k^{4}-0.18403229\times10^{-2}k^{5}+0.36946614\times10^{-2}k^{6} \\
& +0.94039351\times10^{-4}k^{7}-0.39062499\times10^{-3}k^{8}-0.57870370\times10^{-4}k^{9})n \\
& +(-0.46703197\times10^{-2}-0.96819842\times10^{-3}k-0.15962049\times10^{-1}k^{2} \\
& -0.17787226\times10^{-2}k^{3}+0.87911534\times10^{-2}k^{4}-0.39876302\times10^{-3}k^{5} \\
& -0.25300203\times10^{-2}k^{6}-0.65104166\times10^{-3}k^{7}-0.43402777\times10^{-4}k^{8})n^{2} \\
& +(0.67703811\times10^{-3}+0.12057833\times10^{-2}k+0.12190190\times10^{-1}k^{2}-0.24640118\times10^{-4}k^{3} \\
& -0.46748408\times10^{-2}k^{4}-0.53530093\times10^{-3}k^{5}+0.34722222\times10^{-3}k^{6} \\
& +0.57870370\times10^{-4}k^{7})n^{3} \\
& +(0.33632631\times10^{-2}-0.61255560\times10^{-3}k-0.59137414\times10^{-2}k^{2}-0.18593117\times10^{-3}k^{3} \\
& +0.13495551\times10^{-2}k^{4}+0.28754339\times10^{-3}k^{5}+0.72337963\times10^{-5}k^{6})n^{4} \\
& +(-0.30259309\times10^{-2}+0.40237991\times10^{-4}k+0.19541422\times10^{-2}k^{2} \\
& +0.27895326\times10^{-3}k^{3}-0.13563368\times10^{-3}k^{4}-0.21701388\times10^{-4}k^{5})n^{5} \\
& +(0.12293497\times10^{-2}+0.10522912\times10^{-3}k-0.34371835\times10^{-3}k^{2}-0.62391493\times10^{-4}k^{3} \\
& +0.18084490\times10^{-5}k^{4})n^{6} \\
& +(-0.27477122\times10^{-3}-0.40464047\times10^{-4}k+0.27126735\times10^{-4}k^{2} \\
& +0.36168981\times10^{-5}k^{3})n^{7} \\
& +(0.34247503\times10^{-4}+0.52558050\times10^{-5}k-0.67816839\times10^{-6}k^{2})n^{8} \\
& +(-0.22040472\times10^{-5}-0.22605613\times10^{-6}k)n^{9}+0.56514033\times10^{-7}n^{10}
\end{aligned}
$$

$$
\begin{aligned}
X_{5,5}^{n,k} = {} & -0.34772135\times10^{-1}k^{2}+0.37589517\times10^{-1}k^{4}-0.11211208\times10^{-1}k^{6} \\
& +0.13454860\times10^{-2}k^{8}-0.69444444\times10^{-4}k^{10} \\
& +(0.27343749\times10^{-2}+0.10854763\times10^{-1}k^{2}-0.15262519\times10^{-1}k^{4} \\
& +0.36566840\times10^{-2}k^{6}-0.26041666\times10^{-3}k^{8})n \\
& +(-0.46972656\times10^{-2}-0.17721896\times10^{-1}k^{2}+0.11828952\times10^{-1}k^{4} \\
& -0.18337673\times10^{-2}k^{6}+0.86805554\times10^{-4}k^{8})n^{2} \\
& +(0.57915581\times10^{-3}+0.11909315\times10^{-1}k^{2}-0.42860242\times10^{-2}k^{4} \\
& +0.34722222\times10^{-3}k^{6})n^{3} \\
& +(0.34478081\times10^{-2}-0.60770670\times10^{-2}k^{2}+0.11528862\times10^{-2}k^{4} \\
& -0.43402777\times10^{-4}k^{6})n^{4} \\
& +(-0.30405680\times10^{-2}+0.18330891\times10^{-2}k^{2}-0.16276041\times10^{-3}k^{4})n^{5} \\
& +(0.12148030\times10^{-2}-0.33298068\times10^{-3}k^{2}+0.10850694\times10^{-4}k^{4})n^{6} \\
& +(-0.27058919\times10^{-3}+0.32552083\times10^{-4}k^{2})n^{7} \\
& +(0.34586588\times10^{-4}-0.13563368\times10^{-5}k^{2})n^{8}-0.23735894\times10^{-5}n^{9} \\
& +0.67816840\times10^{-7}n^{10}
\end{aligned}
$$

4.3 Generalized Newcomb operators for $\varkappa = 0$

$$\Xi_{0,0}^{n,k} = 0.09999999\times10$$

$$\Xi_{1,0}^{n,k} = 0.09999999\times10k-0.50000000n$$

$$\Xi_{2,0}^{n,k} = 0.62500000k+0.50000000k^2+(-0.37500000-0.50000000k)n+0.12500000n^2$$

$$\Xi_{1,1}^{n,k} = -0.09999999\times10k^2+0.25000000n+0.25000000n^2$$

$$\begin{aligned}\Xi_{3,0}^{n,k} = {} & 0.54166666k+0.62500000k^2+0.16666666k^3\\ & +(-0.35416666-0.68750000k-0.25000000k^2)n+(0.18750000+0.12500000k)n^2\\ & -0.20833333\times10^{-1}n^3\end{aligned}$$

$$\begin{aligned}\Xi_{2,1}^{n,k} = {} & -0.25000000k-0.62500000k^2-0.50000000k^3\\ & +(0.25000000+0.31250000k+0.25000000k^2)n\\ & +(0.62500000\times10^{-1}+0.12500000k)n^2-0.62500000\times10^{-1}n^3\end{aligned}$$

$$\begin{aligned}\Xi_{4,0}^{n,k} = {} & 0.53645833k+0.73697916k^2+0.31250000k^3+0.41666666\times10^{-1}k^4\\ & +(-0.36979166-0.85937500k-0.50000000k^2-0.83333333\times10^{-1}k^3)n\\ & +(0.24739583+0.26562500k+0.62500000\times10^{-1}k^2)n^2\\ & +(-0.46875000\times10^{-1}-0.20833333\times10^{-1}k)n^3+0.26041666\times10^{-2}n^4\end{aligned}$$

$$\begin{aligned}\Xi_{3,1}^{n,k} = {} & -0.38541666k-0.79166666k^2-0.62500000k^3-0.16666666k^4\\ & +(0.32291666+0.61458334k+0.50000000k^2+0.16666666k^3)n\\ & +(-0.41666666\times10^{-1}+0.31250000\times10^{-1}k)n^2\\ & +(-0.62500000\times10^{-1}-0.41666666\times10^{-1}k)n^3+0.10416666\times10^{-1}n^4\end{aligned}$$

$$\begin{aligned}\Xi_{2,2}^{n,k} = {} & 0.10937500k^2+0.25000000k^4-0.31250000\times10^{-1}n\\ & +(-0.78125000\times10^{-1}-0.12500000k^2)n^2-0.31250000\times10^{-1}n^3+0.15625000\times10^{-1}n^4\end{aligned}$$

$$\begin{aligned}\Xi_{5,0}^{n,k} = {} & 0.57135417k+0.87500000k^2+0.46614583k^3+0.10416666k^4+0.83333333\times10^{-2}k^5\\ & +(-0.40859374-0.10625000\times10k-0.77994791k^2-0.21875000k^3\\ & -0.20833333\times10^{-1}k^4)n\\ & +(0.31770833+0.43229166k+0.17187500k^2+0.20833333\times10^{-1}k^3)n^2\\ & +(-0.79427083\times10^{-1}-0.59895833\times10^{-1}k-0.10416666\times10^{-1}k^2)n^3\\ & +(0.78125000\times10^{-2}+0.26041666\times10^{-2}k)n^4-0.26041666\times10^{-3}n^5\end{aligned}$$

$$\begin{aligned}\Xi_{4,1}^{n,k} = {} & -0.53906250k-0.10781249\times10k^2-0.86197916k^3-0.31250000k^4\\ & -0.41666666\times10^{-1}k^5\\ & +(0.43359375+0.10026041\times10k+0.89713541k^2+0.38541666k^3\\ & +0.62500000\times10^{-1}k^4)n\\ & +(-0.15885416-0.14583333k-0.78125000\times10^{-1}k^2-0.20833333\times10^{-1}k^3)n^2\\ & +(-0.45572916\times10^{-1}-0.54687500\times10^{-1}k-0.10416666\times10^{-1}k^2)n^3\\ & +(0.18229166\times10^{-1}+0.78125000\times10^{-2}k)n^4-0.13020833\times10^{-2}n^5\end{aligned}$$

$$\begin{aligned}\Xi_{3,2}^{n,k} = {} & 0.65104166\times10^{-1}k+0.20312500k^2+0.25520833k^3+0.20833333k^4\\ & +0.83333333\times10^{-1}k^5\\ & +(-0.58593750\times10^{-1}-0.14322916k-0.11718750k^2-0.62500000\times10^{-1}k^3\\ & -0.41666666\times10^{-1}k^4)n\\ & +(-0.44270833\times10^{-1}-0.78125000\times10^{-1}k-0.93750000\times10^{-1}k^2\\ & -0.41666666\times10^{-1}k^3)n^2\\ & +(0.10416666\times10^{-1}+0.52083333\times10^{-2}k+0.20833333\times10^{-1}k^2)n^3\\ & +(0.15625000\times10^{-1}+0.52083333\times10^{-2}k)n^4-0.26041666\times10^{-2}n^5\end{aligned}$$

$$\begin{aligned}
\Xi^{n,k}_{6,0} = {} & 0.63697916k+0.10533419\times10k^{2}+0.64746094k^{3}+0.18793403k^{4}\\
&+0.26041666\times10^{-1}k^{5}+0.13888888\times10^{-2}k^{6}\\
&+(-0.46822916-0.13184027\times10k-0.11201172\times10k^{2}-0.40928819k^{3}\\
&-0.67708333\times10^{-1}k^{4}-0.41666666\times10^{-2}k^{5})n\\
&+(0.40568576+0.64095052k+0.33300781k^{2}+0.70312500\times10^{-1}k^{3}\\
&+0.52083333\times10^{-2}k^{4})n^{2}\\
&+(-0.12141927-0.12000868k-0.36458333\times10^{-1}k^{2}-0.34722222\times10^{-2}k^{3})n^{3}\\
&+(0.16167534\times10^{-1}+0.94401041\times10^{-2}k+0.13020833\times10^{-2}k^{2})n^{4}\\
&+(-0.97656250\times10^{-3}-0.26041666\times10^{-3}k)n^{5}+0.21701388\times10^{-4}n^{6}
\end{aligned}$$

$$\begin{aligned}
\Xi^{n,k}_{5,1} = {} & -0.73802083k-0.14867187\times10k^{2}-0.12239583\times10k^{3}-0.50781250k^{4}-0.10416666k^{5}\\
&-0.83333333\times10^{-2}k^{6}\\
&+(0.58802083+0.15304687\times10k+0.14915364\times10k^{2}+0.72916666k^{3}+0.17708333k^{4}\\
&+0.16666666\times10^{-1}k^{5})n\\
&+(-0.31458333-0.42968750k-0.26106770k^{2}-0.83333333\times10^{-1}k^{3}\\
&-0.10416666\times10^{-1}k^{4})n^{2}\\
&+(-0.97656250\times10^{-2}-0.33854166\times10^{-1}k-0.10416666\times10^{-1}k^{2})n^{3}\\
&+(0.22786458\times10^{-1}+0.16927083\times10^{-1}k+0.26041666\times10^{-2}k^{2})n^{4}\\
&+(-0.32552083\times10^{-2}-0.10416666\times10^{-2}k)n^{5}+0.13020833\times10^{-3}n^{6}
\end{aligned}$$

$$\begin{aligned}
\Xi^{n,k}_{4,2} = {} & 0.15885416k+0.43554688k^{2}+0.50553385k^{3}+0.33984375k^{4}+0.13020833k^{5}\\
&+0.20833333\times10^{-1}k^{6}\\
&+(-0.13802083-0.35677083k-0.37988281k^{2}-0.23046875k^{3}-0.98958333\times10^{-1}k^{4}\\
&-0.20833333\times10^{-1}k^{5})n\\
&+(-0.13671874\times10^{-1}-0.53059895\times10^{-1}k-0.83007813\times10^{-1}k^{2}\\
&-0.44270833\times10^{-1}k^{3}-0.52083333\times10^{-2}k^{4})n^{2}\\
&+(0.32877604\times10^{-1}+0.42317708\times10^{-1}k+0.36458333\times10^{-1}k^{2}\\
&+0.10416666\times10^{-1}k^{3})n^{3}\\
&+(0.81380209\times10^{-2}+0.55338541\times10^{-2}k-0.13020833\times10^{-2}k^{2})n^{4}\\
&+(-0.42317708\times10^{-2}-0.13020833\times10^{-2}k)n^{5}+0.32552083\times10^{-3}n^{6}
\end{aligned}$$

$$\begin{aligned}
\Xi^{n,k}_{3,3} = {} & -0.43402770\times10^{-2}k^{2}-0.39930556\times10^{-1}k^{4}-0.27777778\times10^{-1}k^{6}\\
&+(-0.52083334\times10^{-2}+0.16927083\times10^{-1}k^{2}-0.20833333\times10^{-1}k^{4})n\\
&+(-0.34722221\times10^{-2}+0.22135416\times10^{-1}k^{2}+0.20833333\times10^{-1}k^{4})n^{2}\\
&+(0.91145834\times10^{-2}+0.20833333\times10^{-1}k^{2})n^{3}\\
&+(0.30381943\times10^{-2}-0.52083333\times10^{-2}k^{2})n^{4}-0.39062500\times10^{-2}n^{5}\\
&+0.43402778\times10^{-3}n^{6}
\end{aligned}$$

$$\begin{aligned}
\Xi^{n,k}_{7,0} = {} & 0.73277839k+0.12846570\times10k^{2}+0.87345920k^{3}+0.29937065k^{4}\\
&+0.55121527\times10^{-1}k^{5}+0.52083333\times10^{-2}k^{6}+0.19841269\times10^{-3}k^{7}\\
&+(-0.55041077-0.16466471\times10k-0.15512261\times10k^{2}-0.67084418k^{3}-0.14778645k^{4}\\
&-0.16145833\times10^{-1}k^{5}-0.69444443\times10^{-3}k^{6})n\\
&+(0.51830512+0.91026475k+0.56103516k^{2}+0.15809461k^{3}+0.20833333\times10^{-1}k^{4}\\
&+0.10416666\times10^{-2}k^{5})n^{2}\\
&+(-0.17667101-0.20762803k-0.84364148\times10^{-1}k^{2}-0.14322916\times10^{-1}k^{3}\\
&-0.86805555\times10^{-3}k^{4})n^{3}\\
&+(0.28700086\times10^{-1}+0.22460938\times10^{-1}k+0.55338541\times10^{-2}k^{2}\\
&+0.43402778\times10^{-3}k^{3})n^{4}\\
&+(-0.23871527\times10^{-2}-0.11393229\times10^{-2}k-0.13020833\times10^{-3}k^{2})n^{5}\\
&+(0.97656249\times10^{-4}+0.21701388\times10^{-4}k)n^{6}-0.15500992\times10^{-5}n^{7}
\end{aligned}$$

$$
\begin{aligned}
\Xi^{n,k}_{6,1} = {} & -0.10034722\text{x}10k-0.20547960\text{x}10k^2-0.17480034\text{x}10k^3-0.78982205k^4 \\
& -0.19835069k^5-0.26041666\text{x}10^{-1}k^6-0.13888888\text{x}10^{-2}k^7 \\
& +(0.80002170+0.22607313\text{x}10k+0.23626953\text{x}10k^2+0.12662217\text{x}10k^3 \\
& +0.37261284k^4+0.56770833\text{x}10^{-1}k^5+0.34722222\text{x}10^{-2}k^6)n \\
& +(-0.53145617-0.86725260k-0.59619141k^2-0.21950954k^3-0.41666666\text{x}10^{-1}k^4 \\
& -0.31249999\text{x}10^{-2}k^5)n^2 \\
& +(0.53179253\text{x}10^{-1}+0.35536024\text{x}10^{-1}k+0.17306857\text{x}10^{-1}k^2+0.65104166\text{x}10^{-2}k^3 \\
& +0.86805555\text{x}10^{-3}k^4)n^3 \\
& +(0.22406684\text{x}10^{-1}+0.23003472\text{x}10^{-1}k+0.61848958\text{x}10^{-2}k^2 \\
& +0.43402778\text{x}10^{-3}k^3)n^4 \\
& +(-0.54796006\text{x}10^{-2}-0.30924479\text{x}10^{-2}k-0.39062499\text{x}10^{-3}k^2)n^5 \\
& +(0.42317708\text{x}10^{-3}+0.10850694\text{x}10^{-3}k)n^6-0.10850694\text{x}10^{-4}n^7
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{5,2} = {} & 0.31542969k+0.81093750k^2+0.90221354k^3+0.57324219k^4+0.22005208k^5 \\
& +0.46875000\text{x}10^{-1}k^6+0.41666666\text{x}10^{-2}k^7 \\
& +(-0.26884765-0.74348959k-0.85807291k^2-0.55647786k^3-0.23111979k^4 \\
& -0.57812499\text{x}10^{-1}k^5-0.62499999\text{x}10^{-2}k^6)n \\
& +(0.57942708\text{x}10^{-1}+0.58854167\text{x}10^{-1}k+0.48828125\text{x}10^{-3}k^2-0.13346354\text{x}10^{-1}k^3 \\
& +0.10416666\text{x}10^{-2}k^5)n^2 \\
& +(0.47395833\text{x}10^{-1}+0.76334635\text{x}10^{-1}k+0.61035156\text{x}10^{-1}k^2+0.22135416\text{x}10^{-1}k^3 \\
& +0.26041666\text{x}10^{-2}k^4)n^3 \\
& +(-0.24414063\text{x}10^{-2}-0.32552083\text{x}10^{-2}k-0.48828125\text{x}10^{-2}k^2 \\
& -0.13020833\text{x}10^{-2}k^3)n^4 \\
& +(-0.43945313\text{x}10^{-2}-0.24414063\text{x}10^{-2}k-0.13020833\text{x}10^{-3}k^2)n^5 \\
& +(0.74869791\text{x}10^{-3}+0.19531249\text{x}10^{-3}k)n^6-0.32552083\text{x}10^{-4}n^7
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{4,3} = {} & -0.85720489\text{x}10^{-2}k-0.40798610\text{x}10^{-1}k^2-0.80837673\text{x}10^{-1}k^3 \\
& -0.82790798\text{x}10^{-1}k^4-0.54253472\text{x}10^{-1}k^5-0.26041666\text{x}10^{-1}k^6 \\
& -0.69444444\text{x}10^{-2}k^7 \\
& +(0.68901909\text{x}10^{-2}+0.29513889\text{x}10^{-1}k+0.46603733\text{x}10^{-1}k^2+0.34233940\text{x}10^{-1}k^3 \\
& +0.62934029\text{x}10^{-2}k^4+0.26041666\text{x}10^{-2}k^5+0.34722222\text{x}10^{-2}k^6)n \\
& +(0.58593752\text{x}10^{-2}+0.23871528\text{x}10^{-1}k+0.34667969\text{x}10^{-1}k^2+0.30924479\text{x}10^{-1}k^3 \\
& +0.20833333\text{x}10^{-1}k^4+0.52083333\text{x}10^{-2}k^5)n^2 \\
& +(0.41775173\text{x}10^{-2}-0.16276042\text{x}10^{-3}k+0.60221354\text{x}10^{-2}k^2+0.13020833\text{x}10^{-2}k^3 \\
& -0.26041666\text{x}10^{-2}k^4)n^3 \\
& +(-0.17903646\text{x}10^{-2}-0.37977430\text{x}10^{-2}k-0.68359375\text{x}10^{-2}k^2 \\
& -0.13020833\text{x}10^{-2}k^3)n^4 \\
& +(-0.24956597\text{x}10^{-2}-0.48828125\text{x}10^{-3}k+0.65104166\text{x}10^{-3}k^2)n^5 \\
& +(0.81380208\text{x}10^{-3}+0.10850694\text{x}10^{-3}k)n^6-0.54253472\text{x}10^{-4}n^7
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{8,0} = {} & 0.86247984k+0.15842676\text{x}10k^2+0.11626329\text{x}10k^3+0.44837172k^4 \\
& +0.99121094\text{x}10^{-1}k^5+0.12651909\text{x}10^{-1}k^6+0.86805555\text{x}10^{-3}k^7 \\
& +0.24801587\text{x}10^{-4}k^8 \\
& +(-0.65930524-0.20703644\text{x}10k-0.21092420\text{x}10k^2-0.10282796\text{x}10k^3 \\
& -0.27246094k^4-0.40277778\text{x}10^{-1}k^5-0.31249999\text{x}10^{-2}k^6-0.99206349\text{x}10^{-4}k^7)n \\
& +(0.66387454+0.12634195\text{x}10k+0.87946641k^2+0.29859754k^3 \\
& +0.53331163\text{x}10^{-1}k^4+0.48177082\text{x}10^{-2}k^5+0.17361110\text{x}10^{-3}k^6)n^2 \\
& +(-0.25014377-0.33263075k-0.16312663k^2-0.37597656\text{x}10^{-1}k^3 \\
& -0.41232638\text{x}10^{-2}k^4-0.17361110\text{x}10^{-3}k^5)n^3 \\
& +(0.46961127\text{x}10^{-1}+0.44433594\text{x}10^{-1}k+0.14885796\text{x}10^{-1}k^2+0.21158854\text{x}10^{-2}k^3 \\
& +0.10850694\text{x}10^{-3}k^4)n^4 \\
& +(-0.48285589\text{x}10^{-2}-0.31385633\text{x}10^{-2}k-0.65104166\text{x}10^{-3}k^2 \\
& -0.43402777\text{x}10^{-4}k^3)n^5 \\
& +(0.27533636\text{x}10^{-3}+0.11121961\text{x}10^{-3}k+0.10850694\text{x}10^{-4}k^2)n^6 \\
& +(-0.81380208\text{x}10^{-5}-0.15500992\text{x}10^{-5}k)n^7+0.96881200\text{x}10^{-7}n^8
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{7,1} = {} & -0.13608816\text{x}10k-0.28391043\text{x}10k^{2}-0.24933756\text{x}10k^{3}-0.12002821\text{x}10k^{4} \\
& -0.34147135k^{5}-0.57204860\text{x}10^{-1}k^{6}-0.52083333\text{x}10^{-2}k^{7}-0.19841269\text{x}10^{-3}k^{8} \\
& +(0.10896763\text{x}10+0.32758184\text{x}10k+0.36258517\text{x}10k^{2}+0.20885362\text{x}10k^{3} \\
& +0.69444444k^{4}+0.13402777k^{5}+0.13888889\text{x}10^{-1}k^{6}+0.59523808\text{x}10^{-3}k^{7})n \\
& +(-0.83749534-0.15279242\text{x}10k-0.11573567\text{x}10k^{2}-0.47658962k^{3}-0.11154514k^{4} \\
& -0.13802083\text{x}10^{-1}k^{5}-0.69444443\text{x}10^{-3}k^{6})n^{2} \\
& +(0.15638563+0.17897677k+0.96516927\text{x}10^{-1}k^{2}+0.30707464\text{x}10^{-1}k^{3} \\
& +0.52083333\text{x}10^{-2}k^{4}+0.34722222\text{x}10^{-3}k^{5})n^{3} \\
& +(0.13953993\text{x}10^{-1}+0.20453559\text{x}10^{-1}k+0.73513453\text{x}10^{-2}k^{2} \\
& +0.75954860\text{x}10^{-3}k^{3})n^{4} \\
& +(-0.75141058\text{x}10^{-2}-0.57671440\text{x}10^{-2}k-0.13020833\text{x}10^{-2}k^{2} \\
& -0.86805554\text{x}10^{-4}k^{3})n^{5} \\
& +(0.88433159\text{x}10^{-3}+0.40690104\text{x}10^{-3}k+0.43402777\text{x}10^{-4}k^{2})n^{6} \\
& +(-0.43402777\text{x}10^{-4}-0.93005951\text{x}10^{-5}k)n^{7}+0.77504960\text{x}10^{-6}n^{8}
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{6,2} = {} & 0.56482205k+0.14072157\text{x}10k^{2}+0.15327203\text{x}10k^{3}+0.95968153k^{4}+0.37467448k^{5} \\
& +0.90190971\text{x}10^{-1}k^{6}+0.12152778\text{x}10^{-1}k^{7}+0.69444443\text{x}10^{-3}k^{8} \\
& +(-0.47662760-0.13999023\text{x}10k-0.16933160\text{x}10k^{2}-0.11472601\text{x}10k^{3} \\
& -0.49001735k^{4}-0.13333333k^{5}-0.20833333\text{x}10^{-1}k^{6}-0.13888888\text{x}10^{-2}k^{7})n \\
& +(0.19945746+0.32580838k+0.22592773k^{2}+0.93777126\text{x}10^{-1}k^{3} \\
& +0.30598958\text{x}10^{-1}k^{4}+0.70312499\text{x}10^{-2}k^{5}+0.69444443\text{x}10^{-3}k^{6})n^{2} \\
& +(0.45334201\text{x}10^{-1}+0.91780599\text{x}10^{-1}k+0.79915364\text{x}10^{-1}k^{2}+0.33528645\text{x}10^{-1}k^{3} \\
& +0.60763888\text{x}10^{-2}k^{4}+0.34722222\text{x}10^{-3}k^{5})n^{3} \\
& +(-0.15492079\text{x}10^{-1}-0.19639757\text{x}10^{-1}k-0.13590495\text{x}10^{-1}k^{2} \\
& -0.42317708\text{x}10^{-2}k^{3}-0.43402778\text{x}10^{-3}k^{4})n^{4} \\
& +(-0.29839409\text{x}10^{-2}-0.20996094\text{x}10^{-2}k+0.86805554\text{x}10^{-4}k^{3})n^{5} \\
& +(0.11013454\text{x}10^{-2}+0.51540798\text{x}10^{-3}k+0.43402777\text{x}10^{-4}k^{2})n^{6} \\
& +(-0.97656249\text{x}10^{-4}-0.21701388\text{x}10^{-4}k)n^{7}+0.27126735\text{x}10^{-5}n^{8}
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{5,3} = {} & -0.43793403\text{x}10^{-1}k-0.15176866k^{2}-0.23584527k^{3}-0.21013454k^{4}-0.12141927k^{5} \\
& -0.48350694\text{x}10^{-1}k^{6}-0.12152778\text{x}10^{-1}k^{7}-0.13888888\text{x}10^{-2}k^{8} \\
& +(0.38932292\text{x}10^{-1}+0.12560764k+0.17623155k^{2}+0.14203016k^{3} \\
& +0.69444444\text{x}10^{-1}k^{4}+0.25694444\text{x}10^{-1}k^{5}+0.83333333\text{x}10^{-2}k^{6} \\
& +0.13888888\text{x}10^{-2}k^{7})n \\
& +(0.76280382\text{x}10^{-2}+0.33512370\text{x}10^{-1}k+0.53190104\text{x}10^{-1}k^{2}+0.47824436\text{x}10^{-1}k^{3} \\
& +0.28211805\text{x}10^{-1}k^{4}+0.80729166\text{x}10^{-2}k^{5}+0.69444443\text{x}10^{-3}k^{6})n^{2} \\
& +(-0.60384116\text{x}10^{-2}-0.17225477\text{x}10^{-1}k-0.13183594\text{x}10^{-1}k^{2} \\
& -0.87890625\text{x}10^{-2}k^{3}-0.52083333\text{x}10^{-2}k^{4}-0.10416666\text{x}10^{-2}k^{5})n^{3} \\
& +(-0.38845486\text{x}10^{-2}-0.62391493\text{x}10^{-2}k-0.73513453\text{x}10^{-2}k^{2} \\
& -0.22786458\text{x}10^{-2}k^{3})n^{4} \\
& +(-0.73242188\text{x}10^{-3}+0.54796007\text{x}10^{-3}k+0.13020833\text{x}10^{-2}k^{2} \\
& +0.26041666\text{x}10^{-3}k^{3})n^{5} \\
& +(0.85177951\text{x}10^{-3}+0.25499132\text{x}10^{-3}k-0.43402777\text{x}10^{-4}k^{2})n^{6} \\
& +(-0.13020833\text{x}10^{-3}-0.21701388\text{x}10^{-4}k)n^{7}+0.54253472\text{x}10^{-5}n^{8}
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{4,4} = {} & -0.12207033\text{x}10^{-2}k^{2}+0.47268337\text{x}10^{-2}k^{4}+0.54253472\text{x}10^{-2}k^{6} \\
& +0.17361110\text{x}10^{-2}k^{8} \\
& +(-0.97656246\text{x}10^{-3}+0.94943575\text{x}10^{-3}k^{2}-0.28211804\text{x}10^{-2}k^{4} \\
& +0.34722222\text{x}10^{-2}k^{6})n \\
& +(0.40690097\text{x}10^{-3}-0.24549699\text{x}10^{-2}k^{2}-0.11935765\text{x}10^{-2}k^{4} \\
& -0.17361110\text{x}10^{-2}k^{6})n^{2} \\
& +(0.19802516\text{x}10^{-2}-0.24414063\text{x}10^{-3}k^{2}-0.39062500\text{x}10^{-2}k^{4})n^{3} \\
& +(-0.11325411\text{x}10^{-2}-0.25906032\text{x}10^{-2}k^{2}+0.65104166\text{x}10^{-3}k^{4})n^{4} \\
& +(-0.86805553\text{x}10^{-3}+0.13020833\text{x}10^{-2}k^{2})n^{5} \\
& +(0.71885850\text{x}10^{-3}-0.10850694\text{x}10^{-3}k^{2})n^{6}-0.13563368\text{x}10^{-3}n^{7} \\
& +0.67816840\text{x}10^{-5}n^{8}
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{9,0} = {} & 0.10329466\times10k+0.19720044\times10k^{2}+0.15375723\times10k^{3}+0.64850803k^{4} \\
& +0.16339269k^{5}+0.25358073\times10^{-1}k^{6}+0.23799189\times10^{-2}k^{7}+0.12400793\times10^{-3}k^{8} \\
& +0.27557319\times10^{-5}k^{9} \\
& +(-0.80125679-0.26190402\times10k-0.28394803\times10k^{2}-0.15162732\times10k^{3} \\
& -0.45855453k^{4}-0.82535807\times10^{-1}k^{5}-0.87709779\times10^{-2}k^{6}-0.50843254\times10^{-3}k^{7} \\
& -0.12400793\times10^{-4}k^{8})n \\
& +(0.85298239+0.17303805\times10k+0.13214056\times10k^{2}+0.51280200k^{3}+0.11166721k^{4} \\
& +0.13834635\times10^{-1}k^{5}+0.91145833\times10^{-3}k^{6}+0.24801587\times10^{-4}k^{7})n^{2} \\
& +(-0.34838222-0.50899386k-0.28571890k^{2}-0.80396863\times10^{-1}k^{3} \\
& -0.12107566\times10^{-1}k^{4}-0.93315972\times10^{-3}k^{5}-0.28935185\times10^{-4}k^{6})n^{3} \\
& +(0.73151313\times10^{-1}+0.79335077\times10^{-1}k+0.32491048\times10^{-1}k^{2}+0.63499168\times10^{-2}k^{3} \\
& +0.59678818\times10^{-3}k^{4}+0.21701388\times10^{-4}k^{5})n^{4} \\
& +(-0.87844282\times10^{-2}-0.69892035\times10^{-2}k-0.19958495\times10^{-2}k^{2} \\
& -0.24414063\times10^{-3}k^{3}-0.10850694\times10^{-4}k^{4})n^{5} \\
& +(0.62527126\times10^{-3}+0.34812644\times10^{-3}k+0.62391493\times10^{-4}k^{2} \\
& +0.36168981\times10^{-5}k^{3})n^{6} \\
& +(-0.25996455\times10^{-4}-0.91068327\times10^{-5}k-0.77504960\times10^{-6}k^{2})n^{7} \\
& +(0.58128719\times10^{-6}+0.96881200\times10^{-7}k)n^{8}-0.53822888\times10^{-8}n^{9}
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{8,1} = {} & -0.18437574\times10k-0.39189317\times10k^{2}-0.35451197\times10k^{3}-0.17956190\times10k^{4} \\
& -0.55796372k^{5}-0.10884331k^{6}-0.12999132\times10^{-1}k^{7}-0.86805555\times10^{-3}k^{8} \\
& -0.24801587\times10^{-4}k^{9} \\
& +(0.14847212\times10+0.46878128\times10k+0.54438441\times10k^{2}+0.33268540\times10k^{3} \\
& +0.12091773\times10k^{4}+0.27083875k^{5}+0.36642794\times10^{-1}k^{6}+0.27405754\times10^{-2}k^{7} \\
& +0.86805554\times10^{-4}k^{8})n \\
& +(-0.12703921\times10-0.25109947\times10k-0.20545857\times10k^{2}-0.92531738k^{3} \\
& -0.24857584k^{4}-0.39572482\times10^{-1}k^{5}-0.34288194\times10^{-2}k^{6}-0.12400793\times10^{-3}k^{7})n^{2} \\
& +(0.31917967+0.43611924k+0.26332668k^{2}+0.90786403\times10^{-1}k^{3} \\
& +0.18310547\times10^{-1}k^{4}+0.19748264\times10^{-2}k^{5}+0.86805554\times10^{-4}k^{6})n^{3} \\
& +(-0.76741537\times10^{-2}+0.61374227\times10^{-3}k+0.19666882\times10^{-3}k^{2} \\
& -0.86127386\times10^{-3}k^{3}-0.27126735\times10^{-3}k^{4}-0.21701388\times10^{-4}k^{5})n^{4} \\
& +(-0.86117213\times10^{-2}-0.82275390\times10^{-2}k-0.25302462\times10^{-2}k^{2} \\
& -0.29839409\times10^{-3}k^{3}-0.10850694\times10^{-4}k^{4})n^{5} \\
& +(0.14892577\times10^{-2}+0.93858506\times10^{-3}k+0.18174913\times10^{-3}k^{2} \\
& +0.10850694\times10^{-4}k^{3})n^{6} \\
& +(-0.10782877\times10^{-3}-0.42046440\times10^{-4}k-0.38752480\times10^{-5}k^{2})n^{7} \\
& +(0.36814856\times10^{-5}+0.67816839\times10^{-6}k)n^{8}-0.48440599\times10^{-7}n^{9}
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{7,2} = {} & 0.95208935k+0.23373462\times10k^{2}+0.25243061\times10k^{3}+0.15815294\times10k^{4} \\
& +0.63135036k^{5}+0.16384548k^{6}+0.26736110\times10^{-1}k^{7}+0.24801587\times10^{-2}k^{8} \\
& +0.99206349\times10^{-4}k^{9} \\
& +(-0.79980420-0.24711377\times10k-0.30987229\times10k^{2}-0.21734415\times10k^{3} \\
& -0.96277805k^{4}-0.27955729k^{5}-0.51649304\times10^{-1}k^{6}-0.54563491\times10^{-2}k^{7} \\
& -0.24801587\times10^{-3}k^{8})n \\
& +(0.45398395+0.85692196k+0.71162245k^{2}+0.34965549k^{3}+0.11583116k^{4} \\
& +0.26041666\times10^{-1}k^{5}+0.34722222\times10^{-2}k^{6}+0.19841269\times10^{-3}k^{7})n^{2} \\
& +(0.10463654\times10^{-1}+0.57828775\times10^{-1}k+0.68320041\times10^{-1}k^{2}+0.34233940\times10^{-1}k^{3} \\
& +0.75954860\times10^{-2}k^{4}+0.60763888\times10^{-3}k^{5})n^{3} \\
& +(-0.28952365\times10^{-1}-0.41318088\times10^{-1}k-0.27628580\times10^{-1}k^{2} \\
& -0.94943575\times10^{-2}k^{3}-0.15190972\times10^{-2}k^{4}-0.86805554\times10^{-4}k^{5})n^{4} \\
& +(0.24481880\times10^{-3}+0.65782336\times10^{-3}k+0.11718750\times10^{-2}k^{2}+0.43402778\times10^{-3}k^{3} \\
& +0.43402777\times10^{-4}k^{4})n^{5} \\
& +(0.12356228\times10^{-2}+0.77311198\times10^{-3}k+0.10850694\times10^{-3}k^{2})n^{6} \\
& +(-0.18310547\times10^{-3}-0.73629711\times10^{-4}k-0.62003968\times10^{-5}k^{2})n^{7} \\
& +(0.10075644\times10^{-4}+0.19376240\times10^{-5}k)n^{8}-0.19376240\times10^{-6}n^{9}
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n.k}_{6.3} = {} & -0.12793376k-0.39586046k^{2}-0.55265255k^{3}-0.45595431k^{4}-0.24704047k^{5} \\
& -0.92057291\times10^{-1}k^{6}-0.23090277\times10^{-1}k^{7}-0.34722222\times10^{-2}k^{8} \\
& -0.23148148\times10^{-3}k^{9} \\
& +(0.1o2a9093+0.35975207k+0.50252031k^{2}+0.41139458k^{3}+0.21881555k^{4} \\
& +0.82118055\times10^{-1}k^{5}+0.22800925\times10^{-1}k^{6}+0.41666666\times10^{-2}k^{7} \\
& +0.34722222\times10^{-3}k^{8})n \\
& +(-0.13525391\times10^{-1}-0.20227507\times10^{-2}k+0.30662706\times10^{-1}k^{2} \\
& +0.41347475\times10^{-1}k^{3}+0.27289496\times10^{-1}k^{4}+0.87673610\times10^{-2}k^{5} \\
& +0.10416666\times10^{-2}k^{6})n^{2} \\
& +(-0.20912226\times10^{-1}-0.47946506\times10^{-1}k-0.45396141\times10^{-1}k^{2} \\
& -0.27235243\times10^{-1}k^{3}-0.11574074\times10^{-1}k^{4}-0.26909722\times10^{-2}k^{5} \\
& -0.23148148\times10^{-3}k^{6})n^{3} \\
& +(-0.20833333\times10^{-2}-0.35567672\times10^{-2}k-0.47064886\times10^{-2}k^{2} \\
& -0.14829282\times10^{-2}k^{3}+0.21701389\times10^{-3}k^{4}+0.86805554\times10^{-4}k^{5})n^{4} \\
& +(0.10504828\times10^{-2}+0.22800022\times10^{-2}k+0.22406683\times10^{-2}k^{2}+0.65104166\times10^{-3}k^{3} \\
& +0.43402777\times10^{-4}k^{4})n^{5} \\
& +(0.61442056\times10^{-3}+0.16908998\times10^{-3}k-0.13020833\times10^{-3}k^{2} \\
& -0.28935185\times10^{-4}k^{3})n^{6} \\
& +(-0.18581814\times10^{-3}-0.59678819\times10^{-4}k)n^{7} \\
& +(0.16276041\times10^{-4}+0.27126735\times10^{-5}k)n^{8}-0.45211226\times10^{-6}n^{9}
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n.k}_{5.4} = {} & 0.10504829\times10^{-2}k+0.54416235\times10^{-2}k^{2}+0.14114041\times10^{-1}k^{3}+0.21535915\times10^{-1}k^{4} \\
& +0.19581434\times10^{-1}k^{5}+0.11697048\times10^{-1}k^{6}+0.51649305\times10^{-2}k^{7} \\
& +0.17361110\times10^{-2}k^{8}+0.34722222\times10^{-3}k^{9} \\
& +(-0.13119167\times10^{-2}-0.46956382\times10^{-2}k-0.81610788\times10^{-2}k^{2} \\
& -0.84947377\times10^{-2}k^{3}-0.66602918\times10^{-2}k^{4}-0.16167534\times10^{-2}k^{5} \\
& +0.97656249\times10^{-3}k^{6}+0.17361110\times10^{-3}k^{7}-0.17361110\times10^{-3}k^{8})n \\
& +(-0.52354611\times10^{-3}-0.42263455\times10^{-2}k-0.91050889\times10^{-2}k^{2} \\
& -0.10316297\times10^{-1}k^{3}-0.62120226\times10^{-2}k^{4}-0.40147569\times10^{-2}k^{5} \\
& -0.19965278\times10^{-2}k^{6}-0.34722222\times10^{-3}k^{7})n^{2} \\
& +(0.69308808\times10^{-3}-0.93994140\times10^{-3}k-0.53168418\times10^{-3}k^{2}-0.13970269\times10^{-2}k^{3} \\
& -0.22243923\times10^{-2}k^{4}-0.39062499\times10^{-3}k^{5}+0.17361110\times10^{-3}k^{6})n^{3} \\
& +(-0.65239802\times10^{-3}+0.70665149\times10^{-3}k-0.35264756\times10^{-3}k^{2} \\
& +0.80023871\times10^{-3}k^{3}+0.97656250\times10^{-3}k^{4}+0.13020833\times10^{-3}k^{5})n^{4} \\
& +(-0.12817383\times10^{-3}+0.38384331\times10^{-3}k+0.11135525\times10^{-2}k^{2} \\
& +0.16276041\times10^{-3}k^{3}-0.65104166\times10^{-4}k^{4})n^{5} \\
& +(0.46251084\times10^{-3}-0.13563368\times10^{-4}k-0.22243923\times10^{-3}k^{2} \\
& -0.21701388\times10^{-4}k^{3})n^{6} \\
& +(-0.16276041\times10^{-3}-0.18988714\times10^{-4}k+0.10850694\times10^{-4}k^{2})n^{7} \\
& +(0.18988714\times10^{-4}+0.13563368\times10^{-5}k)n^{8}-0.67816839\times10^{-6}n^{9}
\end{aligned}
$$

$$
\begin{aligned}
\Xi_{10,0}^{n,k} = {} & 0.12541313\times10k + 0.24738538\times10k^{2} + 0.20272344\times10k^{3} + 0.91804938k^{4} \\
& + 0.25555501k^{5} + 0.45655404\times10^{-1}k^{6} + 0.52616825\times10^{-2}k^{7} + 0.37874090\times10^{-3}k^{8} \\
& + 0.15500992\times10^{-4}k^{9} + 0.27557319\times10^{-6}k^{10} \\
& + (-0.98501691 - 0.33310774\times10k - 0.38008564\times10k^{2} - 0.21820702\times10k^{3} \\
& - 0.72998835k^{4} - 0.15142390k^{5} - 0.19781720\times10^{-1}k^{6} - 0.15857515\times10^{-2}k^{7} \\
& - 0.71304563\times10^{-4}k^{8} - 0.13778659\times10^{-5}k^{9})n \\
& + (0.10994266\times10 + 0.23506019\times10k + 0.19323537\times10k^{2} + 0.83054125k^{3} \\
& + 0.20867075k^{4} + 0.31815592\times10^{-1}k^{5} + 0.29016565\times10^{-2}k^{6} + 0.14570932\times10^{-3}k^{7} \\
& + 0.31001984\times10^{-5}k^{8})n^{2} \\
& + (-0.48009884 - 0.75606086k - 0.47061253k^{2} - 0.15296009k^{3} - 0.28379086\times10^{-1}k^{4} \\
& - 0.30309606\times10^{-2}k^{5} - 0.17361110\times10^{-3}k^{6} - 0.41335978\times10^{-5}k^{7})n^{3} \\
& + (0.11033359 + 0.13284361k + 0.62912269\times10^{-1}k^{2} + 0.15163562\times10^{-1}k^{3} \\
& + 0.19768608\times10^{-2}k^{4} + 0.13292100\times10^{-3}k^{5} + 0.36168981\times10^{-5}k^{6})n^{4} \\
& + (-0.14948159\times10^{-1} - 0.13767993\times10^{-1}k - 0.48538207\times10^{-2}k^{2} \\
& - 0.82442671\times10^{-3}k^{3} - 0.67816839\times10^{-4}k^{4} - 0.21701389\times10^{-5}k^{5})n^{5} \\
& + (0.12526505\times10^{-2} a\, 0.86189552\times10^{-3}k + 0.21469681\times10^{-3}k^{2} + 0.23057725\times10^{-4}k^{3} \\
& + 0.90422453\times10^{-6}k^{4})n^{6} \\
& + (-0.65499765\times10^{-4} - 0.31922355\times10^{-4}k - 0.50378224\times10^{-5}k^{2} \\
& - 0.25834987\times10^{-6}k^{3})n^{7} \\
& + (0.20748723\times10^{-5} + 0.64183795\times10^{-6}k + 0.48440599\times10^{-7}k^{2})n^{8} \\
& + (-0.36330450\times10^{-7} - 0.53822888\times10^{-8}k)n^{9} + 0.26911444\times10^{-9}n^{10}
\end{aligned}
$$

$$
\begin{aligned}
\Xi_{9,1}^{n,k} = {} & -0.24971396\times10k - 0.54034126\times10k^{2} - 0.50223409\times10k^{3} - 0.26546379\times10k^{4} \\
& - 0.88268681k^{5} - 0.19170215k^{6} - 0.27195457\times10^{-1}k^{7} - 0.24295221\times10^{-2}k^{8} \\
& - 0.12400793\times10^{-3}k^{9} - 0.27557319\times10^{-5}k^{10} \\
& + (0.20230656\times10 + 0.66504307\times10k + 0.80451310\times10k^{2} + 0.51678418\times10k^{3} \\
& + 0.20141879\times10k^{4} + 0.50171440k^{5} + 0.80412687\times10^{-1}k^{6} + 0.80212466\times10^{-2}k^{7} \\
& + 0.45262896\times10^{-3}k^{8} + 0.11022928\times10^{-4}k^{9})n \\
& + (-0.18819387\times10 - 0.39573697\times10k - 0.34487605\times10k^{2} - 0.16714495\times10k^{3} \\
& - 0.49733768k^{4} - 0.93066406\times10^{-1}k^{5} - 0.10664424\times10^{-1}k^{6} - 0.68204364\times10^{-3}k^{7} \\
& - 0.18601190\times10^{-4}k^{8})n^{2} \\
& + (0.56950028 + 0.86650268k + 0.57304879k^{2} + 0.21598714k^{3} + 0.49705222\times10^{-1}k^{4} \\
& + 0.68721064\times10^{-2}k^{5} + 0.52083333\times10^{-3}k^{6} + 0.16534391\times10^{-4}k^{7})n^{3} \\
& + (-0.50390571\times10^{-1} - 0.50232385\times10^{-1}k - 0.25295342\times10^{-1}k^{2} \\
& - 0.82793058\times10^{-2}k^{3} - 0.16524703\times10^{-2}k^{4} - 0.17361110\times10^{-3}k^{5} \\
& - 0.72337963\times10^{-5}k^{6})n^{4} \\
& + (-0.75208310\times10^{-2} - 0.88684081\times10^{-2}k - 0.33457437\times10^{-2}k^{2} \\
& - 0.50907841\times10^{-3}k^{3} - 0.27126735\times10^{-4}k^{4})n^{5} \\
& + (0.21600229\times10^{-2} + 0.16829879\times10^{-2}k + 0.44906051\times10^{-3}k^{2} + 0.48828124\times10^{-4}k^{3} \\
& + 0.18084490\times10^{-5}k^{4})n^{6} \\
& + (-0.21237973\times10^{-3} - 0.11444899\times10^{-3}k - 0.19376240\times10^{-4}k^{2} \\
& - 0.10333994\times10^{-5}k^{3})n^{7} \\
& + (0.10576198\times10^{-4} + 0.35846043\times10^{-5}k + 0.29064360\times10^{-6}k^{2})n^{8} \\
& + (-0.26642329\times10^{-6} - 0.43058310\times10^{-7}k)n^{9} + 0.26911444\times10^{-8}n^{10}
\end{aligned}
$$

$$\Xi^{n,k}_{8,2} = 0.15432477x10k+0.37668870x10k^{2}+0.40641558x10k^{3}+0.25650379x10k^{4}$$
$$+0.10481381x10k^{5}+0.28745218k^{6}+0.52677408x10^{-1}k^{7}+0.61802455x10^{-2}k^{8}$$
$$+0.41852678x10^{-3}k^{9}+0.12400793x10^{-4}k^{10}$$
$$+(-0.12946428x10-0.41740547x10k-0.53964563x10k^{2}-0.39002837x10k^{3}$$
$$-0.17897441x10k^{4}-0.54868198k^{5}-0.11261664k^{6}-0.14826699x10^{-1}k^{7}$$
$$-0.11253720x10^{-2}k^{8}-0.37202380x10^{-4}k^{9})n$$
$$+(0.88705066+0.18235785x10k+0.16494814x10k^{2}+0.87781762k^{3}+0.30782021k^{4}$$
$$+0.73409017x10^{-1}k^{5}+0.11433919x10^{-1}k^{6}+0.10323660x10^{-2}k^{7}$$
$$+0.40302579x10^{-4}k^{8})n^{2}$$
$$+(-0.84595551x10^{-1}-0.79962256x10^{-1}k-0.20364550x10^{-1}k^{2}$$
$$+0.16503228x10^{-2}k^{3}+0.67816795x10^{-4}k^{4}-0.67274304x10^{-3}k^{5}$$
$$-0.17361110x10^{-3}k^{6}-0.12400793x10^{-4}k^{7})n^{3}$$
$$+(-0.38277277x10^{-1}-0.62188975x10^{-1}k-0.44248962x10^{-1}k^{2}$$
$$-0.16710917x10^{-1}k^{3}-0.33264160x10^{-2}k^{4}-0.31738281x10^{-3}k^{5}$$
$$-0.10850694x10^{-4}k^{6})n^{4}$$
$$+(0.53994920x10^{-2}+0.65836588x10^{-2}k+0.42017618x10^{-2}k^{2}+0.13678657x10^{-2}k^{3}$$
$$+0.20345052x10^{-3}k^{4}+0.10850694x10^{-4}k^{5})n^{5}$$
$$+(0.98054674x10^{-3}+0.71122911x10^{-3}k+0.85618758x10^{-4}k^{2}-0.17632378x10^{-4}k^{3}$$
$$-0.27126735x10^{-5}k^{4})n^{6}$$
$$+(-0.26736789x10^{-3}-0.14653281x10^{-3}k-0.22088913x10^{-4}k^{2}$$
$$-0.77504960x10^{-6}k^{3})n^{7}$$
$$+(0.22839743x10^{-4}+0.80290294x10^{-5}k+0.62972780x10^{-6}k^{2})n^{8}$$
$$+(-0.85982065x10^{-6}-0.14532179x10^{-6}k)n^{9}+0.12110150x10^{-7}n^{10}$$

$$\Xi^{n,k}_{7,3} = -0.30049706k-0.87614863k^{2}-0.11560290x10k^{3}-0.91120037k^{4}-0.47678494k^{5}$$
$$-0.17344021k^{6}-0.43988715x10^{-1}k^{7}-0.74280753x10^{-2}k^{8}-0.74404761x10^{-3}k^{9}$$
$$-0.33068783x10^{-4}k^{10}$$
$$+(0.26246797+0.85527629k+0.12066446x10k^{2}+0.99665043k^{3}+0.54149735k^{4}$$
$$+0.20606191k^{5}+0.56532117x10^{-1}k^{6}+0.10809358x10^{-1}k^{7}+0.12648809x10^{-2}k^{8}$$
$$+0.66137566x10^{-4}k^{9})n$$
$$+(-0.87270845x10^{-1}-0.15401746k-0.11042726k^{2}-0.37830042x10^{-1}k^{3}$$
$$-0.33142090x10^{-2}k^{4}+0.71614578x10^{-3}k^{5}-0.39785879x10^{-3}k^{6}$$
$$-0.22321428x10^{-3}k^{7}-0.24801587x10^{-4}k^{8})n^{2}$$
$$+(-0.34885499x10^{-1}-0.84285417x10^{-1}k-0.87155038x10^{-1}k^{2}$$
$$-0.53502061x10^{-1}k^{3}-0.21737557x10^{-1}k^{4}-0.54108796x10^{-2}k^{5}$$
$$-0.69444443x10^{-3}k^{6}-0.33068783x10^{-4}k^{7})n^{3}$$
$$+(0.50561652x10^{-2}+0.80819590x10^{-2}k+0.47403971x10^{-2}k^{2}+0.26697230x10^{-2}k^{3}$$
$$+0.14015480x10^{-2}k^{4}+0.34722222x10^{-3}k^{5}+0.28935185x10^{-4}k^{6})n^{4}$$
$$+(0.22133156x10^{-2}+0.37213361x10^{-2}k+0.30802408x10^{-2}k^{2}+0.10181568x10^{-2}k^{3}$$
$$+0.10850694x10^{-3}k^{4})n^{5}$$
$$+(0.14173719x10^{-3}-0.21927445x10^{-3}k-0.33682363x10^{-3}k^{2}-0.97656249x10^{-4}k^{3}$$
$$-0.72337963x10^{-5}k^{4})n^{6}$$
$$+(-0.19757306x10^{-3}-0.80088459x10^{-4}k+0.31001984x10^{-5}k^{2}$$
$$+0.20667989x10^{-5}k^{3})n^{7}$$
$$+(0.29258122x10^{-4}+0.91068327x10^{-5}k+0.38752480x10^{-6}k^{2})n^{8}$$
$$+(-0.16469803x10^{-5}-0.25834987x10^{-6}k)n^{9}+0.32293733x10^{-7}n^{10}$$

$$
\begin{aligned}
\Xi_{6,4}^{n,k} = {} & 0.94654227\times10^{-2}k+0.38994685\times10^{-1}k^{2}+0.72781712\times10^{-1}k^{3}+0.82432953\times10^{-1}k^{4} \\
& +0.62127516\times10^{-1}k^{5}+0.32540555\times10^{-1}k^{6}+0.12418620\times10^{-1}k^{7} \\
& +0.34939235\times10^{-2}k^{8}+0.65104166\times10^{-3}k^{9}+0.57870370\times10^{-4}k^{10} \\
& +(-0.89011868\times10^{-2}-0.32894786\times10^{-1}k-0.54654722\times10^{-1}k^{2} \\
& -0.53465892\times10^{-1}k^{3}-0.35591578\times10^{-1}k^{4}-0.15816017\times10^{-1}k^{5} \\
& -0.46386719\times10^{-2}k^{6}-0.12948495\times10^{-2}k^{7}-0.39062499\times10^{-3}k^{8} \\
& -0.57870370\times10^{-4}k^{9})n \\
& +(-0.23292824\times10^{-2}-0.11470314\times10^{-1}k-0.22460259\times10^{-1}k^{2} \\
& -0.24700814\times10^{-1}k^{3}-0.16490965\times10^{-1}k^{4}-0.84716796\times10^{-2}k^{5} \\
& -0.32244647\times10^{-2}k^{6}-0.65104166\times10^{-3}k^{7}-0.43402777\times10^{-4}k^{8})n^{2} \\
& +(0.18936723\times10^{-2}+0.37421333\times10^{-2}k+0.48659714\times10^{-2}k^{2}+0.29050475\times10^{-2}k^{3} \\
& +0.53349250\times10^{-3}k^{4}+0.50636574\times10^{-3}k^{5}+0.34722222\times10^{-3}k^{6} \\
& +0.57870370\times10^{-4}k^{7})n^{3} \\
& +(0.39152926\times10^{-3}+0.25138007\times10^{-2}k+0.21700258\times10^{-2}k^{2}+0.20927146\times10^{-2}k^{3} \\
& +0.13495551\times10^{-2}k^{4}+0.28754339\times10^{-3}k^{5}+0.72337963\times10^{-5}k^{6})n^{4} \\
& +(0.86862069\times10^{-4}+0.22469979\times10^{-3}k+0.65205890\times10^{-3}k^{2}+0.18536599\times10^{-4}k^{3} \\
& -0.13563368\times10^{-3}k^{4}-0.21701388\times10^{-4}k^{5})n^{5} \\
& +(0.19446479\times10^{-3}-0.14976218\times10^{-3}k-0.30031557\times10^{-3}k^{2}-0.62391493\times10^{-4}k^{3} \\
& +0.18084490\times10^{-5}k^{4})n^{6} \\
& +(-0.14456289\times10^{-3}-0.18762659\times10^{-4}k+0.27126735\times10^{-4}k^{2} \\
& +0.36168981\times10^{-5}k^{3})n^{7} \\
& +(0.28822157\times10^{-4}+0.52558050\times10^{-5}k-0.67816839\times10^{-6}k^{2})n^{8} \\
& +(-0.22040472\times10^{-5}-0.22605613\times10^{-6}k)n^{9}+0.56514033\times10^{-7}n^{10}
\end{aligned}
$$

$$
\begin{aligned}
\Xi_{5,5}^{n,k} = {} & -0.34830736\times10^{-3}k^{2}+0.63612190\times10^{-3}k^{4}-0.10115561\times10^{-2}k^{6} \\
& -0.39062500\times10^{-3}k^{8}-0.69444444\times10^{-4}k^{10} \\
& +(-0.19531251\times10^{-3}+0.38384331\times10^{-3}k^{2}-0.72258835\times10^{-3}k^{4} \\
& +0.18446179\times10^{-3}k^{6}-0.26041666\times10^{-3}k^{8})n \\
& +(0.26692715\times10^{-3}-0.37434871\times10^{-3}k^{2}+0.13037788\times10^{-2}k^{4} \\
& -0.97656242\times10^{-4}k^{6}+0.86805554\times10^{-4}k^{8})n^{2} \\
& +(0.30788845\times10^{-3}+0.43470587\times10^{-3}k^{2}-0.37977425\times10^{-3}k^{4} \\
& +0.34722222\times10^{-3}k^{6})n^{3} \\
& +(-0.54660374\times10^{-3}-0.55677629\times10^{-3}k^{2}+0.50184461\times10^{-3}k^{4} \\
& -0.43402777\times10^{-4}k^{6})n^{4} \\
& +(0.24753129\times10^{-4}+0.53100585\times10^{-3}k^{2}-0.16276041\times10^{-3}k^{4})n^{5} \\
& +(0.25180392\times10^{-3}-0.22447373\times10^{-3}k^{2}+0.10850694\times10^{-4}k^{4})n^{6} \\
& +(-0.13495550\times10^{-3}+0.32552083\times10^{-4}k^{2})n^{7} \\
& +(0.21023220\times10^{-4}-0.13563368\times10^{-5}k^{2})n^{8}-0.23735894\times10^{-5}n^{9} \\
& +0.67816840\times10^{-7}n^{10}
\end{aligned}
$$

4.4 Generalized Newcomb operators for $\varkappa = 1/2$

$$\Xi_{0,0}^{n,k} = 0.09999999\text{x}10$$

$$\Xi_{1,0}^{n,k} = 0.09999999\text{x}10k-0.50000000n$$

$$\Xi_{2,0}^{n,k} = 0.62500000k+0.50000000k^2+(-0.37500000-0.50000000k)n+0.12500000n^2$$

$$\Xi_{1,1}^{n,k} = -0.09999999\text{x}10k^2-0.75000000n+0.25000000n^2$$

$$\begin{aligned}\Xi_{3,0}^{n,k} = {} & 0.54166666k+0.62500000k^2+0.16666666k^3\\ & +(-0.35416666-0.68750000k-0.25000000k^2)n+(0.18750000+0.12500000k)n^2\\ & -0.20833333\text{x}10^{-1}n^3\end{aligned}$$

$$\begin{aligned}\Xi_{2,1}^{n,k} = {} & -0.62500000k^2-0.50000000k^3+(0.12500000-0.68750000k+0.25000000k^2)n\\ & +(0.56250000+0.12500000k)n^2-0.62500000\text{x}10^{-1}n^3\end{aligned}$$

$$\begin{aligned}\Xi_{4,0}^{n,k} = {} & 0.53645833k+0.73697916k^2+0.31250000k^3+0.41666666\text{x}10^{-1}k^4\\ & +(-0.36979166-0.85937500k-0.50000000k^2-0.83333333\text{x}10^{-1}k^3)n\\ & +(0.24739583+0.26562500k+0.62500000\text{x}10^{-1}k^2)n^2\\ & +(-0.46875000\text{x}10^{-1}-0.20833333\text{x}10^{-1}k)n^3+0.26041666\text{x}10^{-2}n^4\end{aligned}$$

$$\begin{aligned}\Xi_{3,1}^{n,k} = {} & -0.72916666\text{x}10^{-1}k-0.54166666k^2-0.62500000k^3-0.16666666k^4\\ & +(0.13541666-0.26041666k+0.16666666k^3)n+(0.39583333+0.53125000k)n^2\\ & +(-0.18750000-0.41666666\text{x}10^{-1}k)n^3+0.10416666\text{x}10^{-1}n^4\end{aligned}$$

$$\begin{aligned}\Xi_{2,2}^{n,k} = {} & -0.39062500k^2+0.25000000k^4+(-0.15625000+0.09999999\text{x}10k^2)n\\ & +(0.29687500-0.12500000k^2)n^2-0.28125000n^3+0.15625000\text{x}10^{-1}n^4\end{aligned}$$

$$\begin{aligned}\Xi_{5,0}^{n,k} = {} & 0.57135417k+0.87500000k^2+0.46614583k^3+0.10416666k^4+0.83333333\text{x}10^{-2}k^5\\ & +(-0.40859374-0.10625000\text{x}10k-0.77994791k^2-0.21875000k^3\\ & -0.20833333\text{x}10^{-1}k^4)n\\ & +(0.31770833+0.43229166k+0.17187500k^2+0.20833333\text{x}10^{-1}k^3)n^2\\ & +(-0.79427083\text{x}10^{-1}-0.59895833\text{x}10^{-1}k-0.10416666\text{x}10^{-1}k^2)n^3\\ & +(0.78125000\text{x}10^{-2}+0.26041666\text{x}10^{-2}k)n^4-0.26041666\text{x}10^{-3}n^5\end{aligned}$$

$$\begin{aligned}\Xi_{4,1}^{n,k} = {} & -0.13281250k-0.60937500k^2-0.73697916k^3-0.31250000k^4-0.41666666\text{x}10^{-1}k^5\\ & +(0.16796875-0.54687500\text{x}10^{-1}k+0.84635416\text{x}10^{-1}k^2+0.21875000k^3\\ & +0.62500000\text{x}10^{-1}k^4)n\\ & +(0.33593750+0.63541666k+0.17187500k^2-0.20833333\text{x}10^{-1}k^3)n^2\\ & +(-0.24869791-0.17968750k-0.10416666\text{x}10^{-1}k^2)n^3\\ & +(0.39062499\text{x}10^{-1}+0.78125000\text{x}10^{-2}k)n^4-0.13020833\text{x}10^{-2}n^5\end{aligned}$$

$$\begin{aligned}\Xi_{3,2}^{n,k} = {} & -0.28645833\text{x}10^{-1}k-0.26562500k^2-0.11979166k^3+0.20833333k^4\\ & +0.83333333\text{x}10^{-1}k^5\\ & +(0.82031250\text{x}10^{-1}-0.15885416k+0.69531250k^2+0.43750000k^3\\ & -0.41666666\text{x}10^{-1}k^4)n\\ & +(0.26041666\text{x}10^{-2}+0.20312500k-0.34375000k^2-0.41666666\text{x}10^{-1}k^3)n^2\\ & +(-0.34895833-0.11979166k+0.20833333\text{x}10^{-1}k^2)n^3\\ & +(0.78125000\text{x}10^{-1}+0.52083333\text{x}10^{-2}k)n^4-0.26041666\text{x}10^{-2}n^5\end{aligned}$$

$$\begin{aligned}\Xi_{6,0}^{n,k} = {} & 0.63697916k+0.10533419\text{x}10k^2+0.64746094k^3+0.18793403k^4\\ & +0.26041666\text{x}10^{-1}k^5+0.13888888\text{x}10^{-2}k^6\\ & +(-0.46822916-0.13184027\text{x}10k-0.11201172\text{x}10k^2-0.40928819k^3\\ & -0.67708333\text{x}10^{-1}k^4-0.41666666\text{x}10^{-2}k^5)n\\ & +(0.40568576+0.64095052k+0.33300781k^2+0.70312500\text{x}10^{-1}k^3\\ & +0.52083333\text{x}10^{-2}k^4)n^2\\ & +(-0.12141927-0.12000868k-0.36458333\text{x}10^{-1}k^2-0.34722222\text{x}10^{-2}k^3)n^3\\ & +(0.16167534\text{x}10^{-1}+0.94401041\text{x}10^{-2}k+0.13020833\text{x}10^{-2}k^2)n^4\\ & +(-0.97656250\text{x}10^{-3}-0.26041666\text{x}10^{-3}k)n^5+0.21701388\text{x}10^{-4}n^6\end{aligned}$$

$$\Xi^{n,k}_{5,1} = -0.20156250k-0.74973959k^2-0.91145833k^3-0.46614583k^4-0.10416666k^5$$
$$-0.83333333\text{x}10^{-2}k^6$$
$$+(0.21822917+0.13463541k+0.25455729k^2+0.33333334k^3+0.13541666k^4$$
$$+0.16666666\text{x}10^{-1}k^5)n$$
$$+(0.30260416+0.69531250k+0.30143229k^2-0.10416666\text{x}10^{-1}k^4)n^2$$
$$+(-0.30403645-0.32031250k-0.72916666\text{x}10^{-1}k^2)n^3$$
$$+(0.72265625\text{x}10^{-1}+0.37760416\text{x}10^{-1}k+0.26041666\text{x}10^{-2}k^2)n^4$$
$$+(-0.58593749\text{x}10^{-2}-0.10416666\text{x}10^{-2}k)n^5+0.13020833\text{x}10^{-3}n^6$$

$$\Xi^{n,k}_{4,2} = -0.70312498\text{x}10^{-1}k-0.23111979k^2-0.11946614k^3+0.17317708k^4+0.13020833k^5$$
$$+0.20833333\text{x}10^{-1}k^6$$
$$+(0.91145831\text{x}10^{-1}+0.49479163\text{x}10^{-1}k+0.53678385k^2+0.56119791k^3$$
$$+0.67708333\text{x}10^{-1}k^4-0.20833333\text{x}10^{-1}k^5)n$$
$$+(-0.65755208\text{x}10^{-1}+0.51106770\text{x}10^{-1}k-0.33300781k^2-0.21093749k^3$$
$$-0.52083333\text{x}10^{-2}k^4)n^2$$
$$+(-0.26920573-0.28059895k+0.36458333\text{x}10^{-1}k^2+0.10416666\text{x}10^{-1}k^3)n^3$$
$$+(0.14355468+0.47200520\text{x}10^{-1}k-0.13020833\text{x}10^{-2}k^2)n^4$$
$$+(-0.14648437\text{x}10^{-1}-0.13020833\text{x}10^{-2}k)n^5+0.32552083\text{x}10^{-3}n^6$$

$$\Xi^{n,k}_{3,3} = -0.14496528k^2+0.21006944k^4-0.27777778\text{x}10^{-1}k^6$$
$$+(-0.57291666\text{x}10^{-1}+0.65755208k^2-0.27083333k^4)n$$
$$+(0.74652777\text{x}10^{-1}-0.60286458k^2+0.20833333\text{x}10^{-1}k^4)n^2$$
$$+(-0.17317708+0.14583333k^2)n^3+(0.17491319-0.52083333\text{x}10^{-2}k^2)n^4$$
$$-0.19531250\text{x}10^{-1}n^5+0.43402778\text{x}10^{-3}n^6$$

$$\Xi^{n,k}_{7,0} = 0.73277839k+0.12846570\text{x}10k^2+0.87345920k^3+0.29937065k^4$$
$$+0.55121527\text{x}10^{-1}k^5+0.52083333\text{x}10^{-2}k^6+0.19841269\text{x}10^{-3}k^7$$
$$+(-0.55041077-0.16466471\text{x}10k-0.15512261\text{x}10k^2-0.67084418k^3-0.14778645k^4$$
$$-0.16145833\text{x}10^{-1}k^5-0.69444443\text{x}10^{-3}k^6)n$$
$$+(0.51830512+0.91026475k+0.56103516k^2+0.15809461k^3+0.20833333\text{x}10^{-1}k^4$$
$$+0.10416666\text{x}10^{-2}k^5)n^2$$
$$+(-0.17667101-0.20762803k-0.84364148\text{x}10^{-1}k^2-0.14322916\text{x}10^{-1}k^3$$
$$-0.86805555\text{x}10^{-3}k^4)n^3$$
$$+(0.28700086\text{x}10^{-1}+0.22460938\text{x}10^{-1}k+0.55338541\text{x}10^{-2}k^2$$
$$+0.43402778\text{x}10^{-3}k^3)n^4$$
$$+(-0.23871527\text{x}10^{-2}-0.11393229\text{x}10^{-2}k-0.13020833\text{x}10^{-3}k^2)n^5$$
$$+(0.97656249\text{x}10^{-4}+0.21701388\text{x}10^{-4}k)n^6-0.15500992\text{x}10^{-5}n^7$$

$$\Xi^{n,k}_{6,1} = -0.28927951k-0.96104600k^2-0.11653211\text{x}10k^3-0.65961371k^4-0.18793403k^5$$
$$-0.26041666\text{x}10^{-1}k^6-0.13888888\text{x}10^{-2}k^7$$
$$+(0.28927951+0.36125217k+0.51276040k^2+0.52663845k^3+0.24240451k^4$$
$$+0.48437499\text{x}10^{-1}k^5+0.34722222\text{x}10^{-2}k^6)n$$
$$+(0.27427300+0.73561198k+0.39860026k^2+0.25282118\text{x}10^{-1}k^3$$
$$-0.20833333\text{x}10^{-1}k^4-0.31249999\text{x}10^{-2}k^5)n^2$$
$$+(-0.36381293-0.47162543k-0.16758897k^2-0.14322916\text{x}10^{-1}k^3$$
$$+0.86805555\text{x}10^{-3}k^4)n^3$$
$$+(0.11159939+0.86154513\text{x}10^{-1}k+0.16601562\text{x}10^{-1}k^2+0.43402778\text{x}10^{-3}k^3)n^4$$
$$+(-0.13617621\text{x}10^{-1}-0.56966145\text{x}10^{-2}k-0.39062499\text{x}10^{-3}k^2)n^5$$
$$+(0.68359374\text{x}10^{-3}+0.10850694\text{x}10^{-3}k)n^6-0.10850694\text{x}10^{-4}n^7$$

$$\Xi^{n,k}_{5,2} = -0.10449219k-0.24375000k^2-0.97135419\times10^{-1}k^3+0.18261719k^4+0.16796875k^5$$
$$+0.46875000\times10^{-1}k^6+0.41666666\times10^{-2}k^7$$
$$+(0.10712890+0.18489584k+0.59928384k^2+0.62060546k^3+0.15950520k^4$$
$$-0.16145833\times10^{-1}k^5-0.62499999\times10^{-2}k^6)n$$
$$+(-0.13216146-0.10911458k-0.43180338k^2-0.34147135k^3-0.62500000\times10^{-1}k^4$$
$$+0.10416666\times10^{-2}k^5)n^2$$
$$+(-0.22506510-0.31494141k+0.11393227\times10^{-2}k^2+0.42968749\times10^{-1}k^3$$
$$+0.26041666\times10^{-2}k^4)n^3$$
$$+(0.18050130+0.12369791k+0.55338541\times10^{-2}k^2-0.13020833\times10^{-2}k^3)n^4$$
$$+(-0.34667969\times10^{-1}-0.10253906\times10^{-1}k-0.13020833\times10^{-3}k^2)n^5$$
$$+(0.20507812\times10^{-2}+0.19531249\times10^{-3}k)n^6-0.32552083\times10^{-4}n^7$$

$$\Xi^{n,k}_{4,3} = -0.53168403\times10^{-2}k-0.79861110\times10^{-1}k^2+0.37977435\times10^{-2}k^3+0.17762586k^4$$
$$+0.49913194\times10^{-1}k^5-0.26041666\times10^{-1}k^6-0.69444444\times10^{-2}k^7$$
$$+(0.31304253\times10^{-1}-0.57725695\times10^{-1}k+0.43918186k^2+0.20090060k^3$$
$$-0.25412326k^4-0.80729166\times10^{-1}k^5+0.34722222\times10^{-2}k^6)n$$
$$+(-0.91796874\times10^{-1}+0.65538195\times10^{-1}k-0.52783203k^2-0.20865885k^3$$
$$+0.62499999\times10^{-1}k^4+0.52083333\times10^{-2}k^5)n^2$$
$$+(-0.92827690\times10^{-1}-0.50944010\times10^{-1}k+0.25081380k^2+0.42968749\times10^{-1}k^3$$
$$-0.26041666\times10^{-2}k^4)n^3$$
$$+(0.18440755+0.60004339\times10^{-1}k-0.27669270\times10^{-1}k^2-0.13020833\times10^{-2}k^3)n^4$$
$$+(-0.52625868\times10^{-1}-0.56966145\times10^{-2}k+0.65104166\times10^{-3}k^2)n^5$$
$$+(0.34179687\times10^{-2}+0.10850694\times10^{-3}k)n^6-0.54253472\times10^{-4}n^7$$

$$\Xi^{n,k}_{8,0} = 0.86247984k+0.15842676\times10k^2+0.11626329\times10k^3+0.44837172k^4$$
$$+0.99121094\times10^{-1}k^5+0.12651909\times10^{-1}k^6+0.86805555\times10^{-3}k^7$$
$$+0.24801587\times10^{-4}k^8$$
$$+(-0.65930524-0.20703644\times10k-0.21092420\times10k^2-0.10282796\times10k^3$$
$$-0.27246094k^4-0.40277778\times10^{-1}k^5-0.31249999\times10^{-2}k^6-0.99206349\times10^{-4}k^7)n$$
$$+(0.66387454+0.12634195\times10k+0.87946641k^2+0.29859754k^3$$
$$+0.53331163\times10^{-1}k^4+0.48177082\times10^{-2}k^5+0.17361110\times10^{-3}k^6)n^2$$
$$+(-0.25014377-0.33263075k-0.16312663k^2-0.37597656\times10^{-1}k^3$$
$$-0.41232638\times10^{-2}k^4-0.17361110\times10^{-3}k^5)n^3$$
$$+(0.46961127\times10^{-1}+0.44433594\times10^{-1}k+0.14885796\times10^{-1}k^2+0.21158854\times10^{-2}k^3$$
$$+0.10850694\times10^{-3}k^4)n^4$$
$$+(-0.48285589\times10^{-2}-0.31385633\times10^{-2}k-0.65104166\times10^{-3}k^2$$
$$-0.43402777\times10^{-4}k^3)n^5$$
$$+(0.27533636\times10^{-3}+0.11121961\times10^{-3}k+0.10850694\times10^{-4}k^2)n^6$$
$$+(-0.81380208\times10^{-5}-0.15500992\times10^{-5}k)n^7+0.96881200\times10^{-7}n^8$$

$$\Xi^{n,k}_{7,1} = -0.40541295k-0.12590913\times10k^2-0.15221842\times10k^3-0.91838108k^4-0.30240885k^5$$
$$-0.55121527\times10^{-1}k^6-0.52083333\times10^{-2}k^7-0.19841269\times10^{-3}k^8$$
$$+(0.38733259+0.66123513k+0.89233398k^2+0.82714302k^3+0.40494791k^4$$
$$+0.10173611k^5+0.12499999\times10^{-1}k^6+0.59523808\times10^{-3}k^7)n$$
$$+(0.23926246+0.75190429k+0.46227213k^2+0.38167317\times10^{-1}k^3$$
$$-0.36024305\times10^{-1}k^4-0.96354165\times10^{-2}k^5-0.69444443\times10^{-3}k^6)n^2$$
$$+(-0.43142904-0.64198676k-0.29117838k^2-0.44813367\times10^{-1}k^3$$
$$+0.34722222\times10^{-3}k^5)n^3$$
$$+(0.15962456+0.15462239k+0.45762803\times10^{-1}k^2+0.42317708\times10^{-2}k^3)n^4$$
$$+(-0.25146484\times10^{-1}-0.15597873\times10^{-1}k-0.26041666\times10^{-2}k^2$$
$$-0.86805554\times10^{-4}k^3)n^5$$
$$+(0.18934461\times10^{-2}+0.66731770\times10^{-3}k+0.43402777\times10^{-4}k^2)n^6$$
$$+(-0.65104166\times10^{-4}-0.93005951\times10^{-5}k)n^7+0.77504960\times10^{-6}n^8$$

$$
\begin{aligned}
\Xi^{n,k}_{6,2} = {} & -0.13986545k-0.27012804k^2-0.68842230x10^{-1}k^3+0.22921278k^4+0.21842448k^5 \\
& +0.77690972x10^{-1}k^6+0.12152778x10^{-1}k^7+0.69444443x10^{-3}k^8 \\
& +(0.12805989+0.31871744k+0.73448351k^2+0.71732314k^3+0.23133680k^4 \\
& -0.41666662x10^{-2}k^5-0.12499999x10^{-1}k^6-0.13888888x10^{-2}k^7)n \\
& +(-0.21265191-0.30752495k-0.61684570k^2-0.49997287k^3-0.14127604k^4 \\
& -0.96354165x10^{-2}k^5+0.69444443x10^{-3}k^6)n^2 \\
& +(-0.18402778-0.30665690k-0.27669270x10^{-2}k^2+0.75195311x10^{-1}k^3 \\
& +0.16493055x10^{-1}k^4+0.34722222x10^{-3}k^5)n^3 \\
& +(0.21269802+0.19845920k+0.31982422x10^{-1}k^2-0.42317708x10^{-2}k^3 \\
& -0.43402778x10^{-3}k^4)n^4 \\
& +(-0.57345919x10^{-1}-0.31005859x10^{-1}k-0.26041666x10^{-2}k^2 \\
& +0.86805554x10^{-4}k^3)n^5 \\
& +(0.58539496x10^{-2}+0.15570746x10^{-2}k+0.43402777x10^{-4}k^2)n^6 \\
& +(-0.22786458x10^{-3}-0.21701388x10^{-4}k)n^7+0.27126735x10^{-5}n^8
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{5,3} = {} & -0.16449651x10^{-1}k-0.29698350x10^{-1}k^2+0.53705513x10^{-1}k^3+0.17463107k^4 \\
& +0.73893228x10^{-1}k^5-0.17100694x10^{-1}k^6-0.12152778x10^{-1}k^7 \\
& -0.13888888x10^{-2}k^8 \\
& +(0.27213540x10^{-1}+0.25347219x10^{-1}k+0.30627712k^2+0.19704318k^3 \\
& -0.21050347k^4-0.13576388k^5-0.12499999x10^{-1}k^6+0.13888888x10^{-2}k^7)n \\
& +(-0.12225477-0.85139970x10^{-1}k-0.46227213k^2-0.30894639k^3 \\
& +0.36024305x10^{-1}k^4+0.28906249x10^{-1}k^5+0.69444443x10^{-3}k^6)n^2 \\
& +(-0.21175130x10^{-1}-0.17876519x10^{-1}k+0.29117838k^2+0.13444010k^3 \\
& -0.10416666x10^{-2}k^5)n^3 \\
& +(0.16554904+0.11078559k-0.45762803x10^{-1}k^2-0.12695312x10^{-1}k^3)n^4 \\
& +(-0.80322264x10^{-1}-0.27772352x10^{-1}k+0.26041666x10^{-2}k^2 \\
& +0.26041666x10^{-3}k^3)n^5 \\
& +(0.10780165x10^{-1}+0.15570746x10^{-2}k-0.43402777x10^{-4}k^2)n^6 \\
& +(-0.45572916x10^{-3}-0.21701388x10^{-4}k)n^7+0.54253472x10^{-5}n^8
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{4,4} = {} & -0.50699868x10^{-1}k^2+0.13233100k^4-0.36241319x10^{-1}k^6+0.17361110x10^{-2}k^8 \\
& +(-0.32226563x10^{-1}+0.35229491k^2-0.30664062k^4+0.31249999x10^{-1}k^6)n \\
& +(0.12125651x10^{-1}-0.52524143k^2+0.17588975k^4-0.17361110x10^{-2}k^6)n^2 \\
& +(-0.36865234x10^{-1}+0.33178710k^2-0.24739583x10^{-1}k^4)n^3 \\
& +(0.13102891-0.93736436x10^{-1}k^2+0.65104166x10^{-3}k^4)n^4 \\
& +(-0.86588541x10^{-1}+0.65104166x10^{-2}k^2)n^5 \\
& +(0.13088650x10^{-1}-0.10850694x10^{-3}k^2)n^6-0.56966145x10^{-3}n^7 \\
& +0.67816840x10^{-5}n^8
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{9,0} = {} & 0.10329466x10k+0.19720044x10k^2+0.15375723x10k^3+0.64850803k^4 \\
& +0.16339269k^5+0.25358073x10^{-1}k^6+0.23799189x10^{-2}k^7+0.12400793x10^{-3}k^8 \\
& +0.27557319x10^{-5}k^9 \\
& +(-0.80125679-0.26190402x10k-0.28394803x10k^2-0.15162732x10k^3 \\
& -0.45855453k^4-0.82535807x10^{-1}k^5-0.87709779x10^{-2}k^6-0.50843254x10^{-3}k^7 \\
& -0.12400793x10^{-4}k^8)n \\
& +(0.85298239+0.17303805x10k+0.13214056x10k^2+0.51280200k^3+0.11166721k^4 \\
& +0.13834635x10^{-1}k^5+0.91145833x10^{-3}k^6+0.24801587x10^{-4}k^7)n^2 \\
& +(-0.34838222-0.50899386k-0.28571890k^2-0.80396863x10^{-1}k^3 \\
& -0.12107566x10^{-1}k^4-0.93315972x10^{-3}k^5-0.28935185x10^{-4}k^6)n^3 \\
& +(0.73151313x10^{-1}+0.79335077x10^{-1}k+0.32491048x10^{-1}k^2+0.63499168x10^{-2}k^3 \\
& +0.59678818x10^{-3}k^4+0.21701388x10^{-4}k^5)n^4 \\
& +(-0.87844282x10^{-2}-0.69892035x10^{-2}k-0.19958495x10^{-2}k^2 \\
& -0.24414063x10^{-3}k^3-0.10850694x10^{-4}k^4)n^5 \\
& +(0.62527126x10^{-3}+0.34812644x10^{-3}k+0.62391493x10^{-4}k^2 \\
& +0.36168981x10^{-5}k^3)n^6 \\
& +(-0.25996455x10^{-4}-0.91068327x10^{-5}k-0.77504960x10^{-6}k^2)n^7 \\
& +(0.58128719x10^{-6}+0.96881200x10^{-7}k)n^8-0.53822888x10^{-8}n^9
\end{aligned}
$$

$$
\begin{aligned}
\Xi_{8,1}^{n,k} = {} & -0.56139527k-0.16707817\times10k^{2}-0.20165660\times10k^{3}-0.12717203\times10k^{4} \\
& -0.46150105k^{5}-0.99728731\times10^{-1}k^{6}-0.12651909\times10^{-1}k^{7}-0.86805555\times10^{-3}k^{8} \\
& -0.24801587\times10^{-4}k^{9} \\
& +(0.52150233+0.10734020\times10k+0.14445412\times10k^{2}+0.12794175\times10k^{3} \\
& +0.65118036k^{4}+0.18746202k^{5}+0.30219184\times10^{-1}k^{6}+0.25421627\times10^{-2}k^{7} \\
& +0.86805554\times10^{-4}k^{8})n \\
& +(0.18705260+0.72861559k+0.47845187k^{2}+0.22192384\times10^{-1}k^{3} \\
& -0.64331055\times10^{-1}k^{4}-0.21603732\times10^{-1}k^{5}-0.27343749\times10^{-2}k^{6} \\
& -0.12400793\times10^{-3}k^{7})n^{2} \\
& +(-0.50829971-0.83749457k-0.44534572k^{2}-0.92373317\times10^{-1}k^{3} \\
& -0.40418836\times10^{-2}k^{4}+0.93315972\times10^{-3}k^{5}+0.86805554\times10^{-4}k^{6})n^{3} \\
& +(0.21922200+0.24754842k+0.94245062\times10^{-1}k^{2}+0.14221191\times10^{-1}k^{3} \\
& +0.59678818\times10^{-3}k^{4}-0.21701388\times10^{-4}k^{5})n^{4} \\
& +(-0.41489325\times10^{-1}-0.32682291\times10^{-1}k-0.82919650\times10^{-2}k^{2} \\
& -0.73242187\times10^{-3}k^{3}-0.10850694\times10^{-4}k^{4})n^{5} \\
& +(0.40473089\times10^{-2}+0.21158854\times10^{-2}k+0.31195746\times10^{-3}k^{2} \\
& +0.10850694\times10^{-4}k^{3})n^{6} \\
& +(-0.20819770\times10^{-3}-0.63747829\times10^{-4}k-0.38752480\times10^{-5}k^{2})n^{7} \\
& +(0.52315848\times10^{-5}+0.67816839\times10^{-6}k)n^{8}-0.48440599\times10^{-7}n^{9}
\end{aligned}
$$

$$
\begin{aligned}
\Xi_{7,2}^{n,k} = {} & -0.17906843k-0.30151559k^{2}-0.24852917\times10^{-1}k^{3}+0.31327312k^{4}+0.29335123k^{5} \\
& +0.11827257k^{6}+0.24305555\times10^{-1}k^{7}+0.24801587\times10^{-2}k^{8}+0.99206349\times10^{-4}k^{9} \\
& +(0.15333436+0.46947370k+0.92522175k^{2}+0.85197344k^{3}+0.30008002k^{4} \\
& +0.56423619\times10^{-2}k^{5}-0.19531250\times10^{-1}k^{6}-0.40674603\times10^{-2}k^{7} \\
& -0.24801587\times10^{-3}k^{8})n \\
& +(-0.31570192-0.56925533k-0.89899766k^{2}-0.71672363k^{3}-0.24636502k^{4} \\
& -0.32031249\times10^{-1}k^{5}+0.19841269\times10^{-3}k^{7})n^{2} \\
& +(-0.13274923-0.25792915k+0.35618760\times10^{-1}k^{2}+0.12451171k^{3} \\
& +0.40364583\times10^{-1}k^{4}+0.37326388\times10^{-2}k^{5})n^{3} \\
& +(0.24361978+0.27223985k+0.67450628\times10^{-1}k^{2}-0.48285589\times10^{-2}k^{3} \\
& -0.23871527\times10^{-2}k^{4}-0.86805554\times10^{-4}k^{5})n^{4} \\
& +(-0.83468288\times10^{-1}-0.61611599\times10^{-1}k-0.10904948\times10^{-1}k^{2} \\
& +0.43402777\times10^{-4}k^{4})n^{5} \\
& +(0.11752658\times10^{-1}+0.53575303\times10^{-2}k+0.49913194\times10^{-3}k^{2})n^{6} \\
& +(-0.75547959\times10^{-3}-0.18213665\times10^{-3}k-0.62003968\times10^{-5}k^{2})n^{7} \\
& +(0.20926338\times10^{-4}+0.19376240\times10^{-5}k)n^{8}-0.19376240\times10^{-6}n^{9}
\end{aligned}
$$

$$
\begin{aligned}
\Xi_{6,3}^{n,k} = {} & -0.17419434\times10^{-1}k+0.14979383\times10^{-1}k^{2}+0.12900435k^{3}+0.20542263k^{4} \\
& +0.92477757\times10^{-1}k^{5}-0.10026041\times10^{-1}k^{6}-0.15798610\times10^{-1}k^{7} \\
& -0.34722222\times10^{-2}k^{8}-0.23148148\times10^{-3}k^{9} \\
& +(0.19708250\times10^{-1}+0.59811745\times10^{-1}k+0.25778559k^{2}+0.12659912k^{3} \\
& -0.22177689k^{4}-0.17660590k^{5}-0.35011573\times10^{-1}k^{6}+0.34722222\times10^{-3}k^{8})n \\
& +(-0.15421007-0.20729619k-0.50510389k^{2}-0.35326514k^{3}+0.84092881\times10^{-2}k^{4} \\
& +0.47569443\times10^{-1}k^{5}+0.72916666\times10^{-2}k^{6})n^{2} \\
& +(0.43597866\times10^{-1}+0.74609374\times10^{-1}k+0.35487241k^{2}+0.22102864k^{3} \\
& +0.24233217\times10^{-1}k^{4}-0.37326388\times10^{-2}k^{5}-0.23148148\times10^{-3}k^{6})n^{3} \\
& +(0.14674208+0.12477982k-0.56057400\times10^{-1}k^{2}-0.36313657\times10^{-1}k^{3} \\
& -0.23871527\times10^{-2}k^{4}+0.86805554\times10^{-4}k^{5})n^{4} \\
& +(-0.10002373-0.58633083\times10^{-1}k+0.16872829\times10^{-2}k^{2}+0.19531249\times10^{-2}k^{3} \\
& +0.43402777\times10^{-4}k^{4})n^{5} \\
& +(0.20859104\times10^{-1}+0.68585431\times10^{-2}k-0.28935185\times10^{-4}k^{3})n^{6} \\
& +(-0.16425239\times10^{-2}-0.25499132\times10^{-3}k)n^{7} \\
& +(0.48828124\times10^{-4}+0.27126735\times10^{-5}k)n^{8}-0.45211226\times10^{-6}n^{9}
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{5,4} = {} & 0.59875485\times10^{-2}k-0.14686415\times10^{-1}k^{2}+0.15063476\times10^{-1}k^{3}+0.10451660k^{4} \\
& +0.15783691\times10^{-1}k^{5}-0.33875868\times10^{-1}k^{6}-0.69878471\times10^{-2}k^{7} \\
& +0.17361110\times10^{-2}k^{8}+0.34722222\times10^{-3}k^{9} \\
& +(0.64734564\times10^{-2}-0.43025716\times10^{-1}k+0.21193169k^{2}+0.50872123\times10^{-1}k^{3} \\
& -0.27092895k^{4}-0.67805988\times10^{-1}k^{5}+0.33094618\times10^{-1} \quad .71180555\times10^{-2}k^{7} \\
& -0.17361110\times10^{-3}k^{8})n \\
& +(-0.68204752\times10^{-1}+0.40749783\times10^{-1}k-0.39575602k^{2}-0.12715115k^{3} \\
& +0.19061957k^{4}+0.44162326\times10^{-1}k^{5}-0.54687499\times10^{-2}k^{6}-0.34722222\times10^{-3}k^{7})n^{2} \\
& +(0.31604003\times10^{-1}+0.40378141\times10^{-2}k+0.34057345k^{2}+0.84540472\times10^{-1}k^{3} \\
& -0.48448350\times10^{-1}k^{4}-0.55989583\times10^{-2}k^{5}+0.17361110\times10^{-3}k^{6})n^{3} \\
& +(0.98319498\times10^{-1}+0.20753309\times10^{-1}k-0.13812934k^{2}-0.23613823\times10^{-1}k^{3} \\
& +0.35807291\times10^{-2}k^{4}+0.13020833\times10^{-3}k^{5})n^{4} \\
& +(-0.94095187\times10^{-1}-0.22714572\times10^{-1}k+0.19505479\times10^{-1}k^{2} \\
& +0.14648437\times10^{-2}k^{3}-0.65104166\times10^{-4}k^{4})n^{5} \\
& +(0.26517741\times10^{-1}+0.32687716\times10^{-2}k-0.87348090\times10^{-3}k^{2} \\
& -0.21701388\times10^{-4}k^{3})n^{6} \\
& +(-0.23735894\times10^{-2}-0.12749565\times10^{-3}k
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{6,3} = {} & -0.17419434\times10^{-1}k+0.14979383\times10^{-1}k^{2}+0.12900435k^{3}+0.20542263k^{4} \\
& +0.92477757\times10^{-1}k^{5}-0.10026041\times10^{-1}k^{6}-0.15798610\times10^{-1}k^{7} \\
& -0.34722222\times10^{-2}k^{8}-0.23148148\times10^{-3}k^{9} \\
& +(0.19708250\times10^{-1}+0.59811745\times10^{-1}k+0.25778559k^{2}+0.12659912k^{3} \\
& -0.22177689k^{4}-0.17660590k^{5}-0.35011573\times10^{-1}k^{6}+0.34722222\times10^{-3}k^{8})n \\
& +(-0.15421007-0.20729619k-0.50510389k^{2}-0.35326514k^{3}+0.84092881\times10^{-2}k^{4} \\
& +0.47569443\times10^{-1}k^{5}+0.72916666\times10^{-2}k^{6})n^{2} \\
& +(0.43597866\times10^{-1}+0.74609374\times10^{-1}k+0.35487241k^{2}+0.22102864k^{3} \\
& +0.24233217\times10^{-1}k^{4}-0.37326388\times10^{-2}k^{5}-0.23148148\times10^{-3}k^{6})n^{3} \\
& +(0.14674208+0.12477982k-0.56057400\times10^{-1}k^{2}-0.36313657\times10^{-1}k^{3} \\
& -0.23871527\times10^{-2}k^{4}+0.86805554\times10^{-4}k^{5})n^{4} \\
& +(-0.10002373-0.58633083\times10^{-1}k+0.16872829\times10^{-2}k^{2}+0.19531249\times10^{-2}k^{3} \\
& +0.43402777\times10^{-4}k^{4})n^{5} \\
& +(0.20859104\times10^{-1}+0.68585431\times10^{-2}k-0.28935185\times10^{-4}k^{3})n^{6} \\
& +(-0.16425239\times10^{-2}-0.25499132\times10^{-3}k)n^{7} \\
& +(0.48828124\times10^{-4}+0.27126735\times10^{-5}k)n^{8}-0.45211226\times10^{-6}n^{9}
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{5,4} = {} & 0.59875485\times10^{-2}k-0.14686415\times10^{-1}k^{2}+0.15063476\times10^{-1}k^{3}+0.10451660k^{4} \\
& +0.15783691\times10^{-1}k^{5}-0.33875868\times10^{-1}k^{6}-0.69878471\times10^{-2}k^{7} \\
& +0.17361110\times10^{-2}k^{8}+0.34722222\times10^{-3}k^{9} \\
& +(0.64734564\times10^{-2}-0.43025716\times10^{-1}k+0.21193169k^{2}+0.50872123\times10^{-1}k^{3} \\
& -0.27092895k^{4}-0.67805988\times10^{-1}k^{5}+0.33094618\times10^{-1}k^{6}+0.71180555\times10^{-2}k^{7} \\
& -0.17361110\times10^{-3}k^{8})n \\
& +(-0.68204752\times10^{-1}+0.40749783\times10^{-1}k-0.39575602k^{2}-0.12715115k^{3} \\
& +0.19061957k^{4}+0.44162326\times10^{-1}k^{5}-0.54687499\times10^{-2}k^{6}-0.34722222\times10^{-3}k^{7})n^{2} \\
& +(0.31604003\times10^{-1}+0.40378141\times10^{-2}k+0.34057345k^{2}+0.84540472\times10^{-1}k^{3} \\
& -0.48448350\times10^{-1}k^{4}-0.55989583\times10^{-2}k^{5}+0.17361110\times10^{-3}k^{6})n^{3} \\
& +(0.98319498\times10^{-1}+0.20753309\times10^{-1}k-0.13812934k^{2}-0.23613823\times10^{-1}k^{3} \\
& +0.35807291\times10^{-2}k^{4}+0.13020833\times10^{-3}k^{5})n^{4} \\
& +(-0.94095187\times10^{-1}-0.22714572\times10^{-1}k+0.19505479\times10^{-1}k^{2} \\
& +0.14648437\times10^{-2}k^{3}-0.65104166\times10^{-4}k^{4})n^{5} \\
& +(0.26517741\times10^{-1}+0.32687716\times10^{-2}k-0.87348090\times10^{-3}k^{2} \\
& -0.21701388\times10^{-4}k^{3})n^{6} \\
& +(-0.23735894\times10^{-2}-0.12749565\times10^{-3}k+0.10850694\times10^{-4}k^{2})n^{7} \\
& +(0.73242187\times10^{-4}+0.13563368\times10^{-5}k)n^{8}-0.67816839\times10^{-6}n^{9}
\end{aligned}
$$

$$\Xi^{n,k}_{10,0} = 0.12541313x10k+0.24738538x10k^2+0.20272344x10k^3+0.91804938k^4$$
$$+0.25555501k^5+0.45655404x10^{-1}k^6+0.52616825x10^{-2}k^7+0.37874090x10^{-3}k^8$$
$$+0.15500992x10^{-4}k^9+0.27557319x10^{-6}k^{10}$$
$$+(-0.98501691-0.33310774x10k-0.38008564x10k^2-0.21820702x10k^3$$
$$-0.72998835k^4-0.15142390k^5-0.19781720x10^{-1}k^6-0.15857515x10^{-2}k^7$$
$$-0.71304563x10^{-4}k^8-0.13778659x10^{-5}k^9)n$$
$$+(0.10994266x10+0.23506019x10k+0.19323537x10k^2+0.83054125k^3$$
$$+0.20867075k^4+0.31815592x10^{-1}k^5+0.29016565x10^{-2}k^6+0.14570932x10^{-3}k^7$$
$$+0.31001984x10^{-5}k^8)n^2$$
$$+(-0.48009884-0.75606086k-0.47061253k^2-0.15296009k^3-0.28379086x10^{-1}k^4$$
$$-0.30309606x10^{-2}k^5-0.17361110x10^{-3}k^6-0.41335978x10^{-5}k^7)n^3$$
$$+(0.11033359+0.13284361k+0.62912269x10^{-1}k^2+0.15163562x10^{-1}k^3$$
$$+0.19768608x10^{-2}k^4+0.13292100x10^{-3}k^5+0.36168981x10^{-5}k^6)n^4$$
$$+(-0.14948159x10^{-1}-0.13767993x10^{-1}k-0.48538207x10^{-2}k^2$$
$$-0.82442671x10^{-3}k^3-0.67816839x10^{-4}k^4-0.21701389x10^{-5}k^5)n^5$$
$$+(0.12526505x10^{-2}+0.86189552x10^{-3}k+0.21469681x10^{-3}k^2+0.23057725x10^{-4}k^3$$
$$+0.90422453x10^{-6}k^4)n^6$$
$$+(-0.65499765x10^{-4}-0.31922355x10^{-4}k-0.50378224x10^{-5}k^2$$
$$-0.25834987x10^{-6}k^3)n^7$$
$$+(0.20748723x10^{-5}+0.64183795x10^{-6}k+0.48440599x10^{-7}k^2)n^8$$
$$+(-0.36330450x10^{-7}-0.53822888x10^{-8}k)n^9+0.26911444x10^{-9}n^{10}$$

$$\Xi^{n,k}_{9,1} = -0.77218005k-0.22348773x10k^2-0.26970751x10k^3-0.17578945x10k^4$$
$$-0.68444462k^5-0.16639833k^6-0.25459345x10^{-1}k^7-0.23799189x10^{-2}k^8$$
$$-0.12400793x10^{-3}k^9-0.27557319x10^{-5}k^{10}$$
$$+(0.70445515+0.16472220x10k+0.22423792x10k^2+0.19486497x10k^3$$
$$+0.10208943x10k^4+0.32203776k^5+0.61510778x10^{-1}k^6+0.69547784x10^{-2}k^7$$
$$+0.42782738x10^{-3}k^8+0.11022928x10^{-4}k^9)n$$
$$+(0.10511559+0.63983387k+0.41941436k^2-0.45974851x10^{-1}k^3-0.11821441k^4$$
$$-0.43153211x10^{-1}k^5-0.71922019x10^{-2}k^6-0.58283729x10^{-3}k^7$$
$$-0.18601190x10^{-4}k^8)n^2$$
$$+(-0.59466181-0.10621784x10k-0.63267086k^2-0.15780571k^3$$
$$-0.11872467x10^{-1}k^4+0.17071759x10^{-2}k^5+0.34722222x10^{-3}k^6$$
$$+0.16534391x10^{-4}k^7)n^3$$
$$+(0.29367545+0.37126555k+0.16760287k^2+0.33550121x10^{-1}k^3$$
$$+0.26878074x10^{-2}k^4-0.72337963x10^{-5}k^6)n^4$$
$$+(-0.64139076x10^{-1}-0.59579128x10^{-1}k-0.19533623x10^{-1}k^2$$
$$-0.27117693x10^{-2}k^3-0.13563368x10^{-3}k^4)n^5$$
$$+(0.75392546x10^{-2}+0.50439905x10^{-2}k+0.11218035x10^{-2}k^2+0.92230903x10^{-4}k^3$$
$$+0.18084490x10^{-5}k^4)n^6$$
$$+(-0.50399214x10^{-3}-0.22876880x10^{-3}k-0.30226934x10^{-4}k^2$$
$$-0.10333994x10^{-5}k^3)n^7$$
$$+(0.18907981x10^{-4}+0.51347036x10^{-5}k+0.29064360x10^{-6}k^2)n^8$$
$$+(-0.36330450x10^{-6}-0.43058310x10^{-7}k)n^9+0.26911444x10^{-8}n^{10}$$

$$
\begin{aligned}
\Xi_{8,2}^{n,k} = \; & -0.22304688k-0.33130872k^2+0.48596089\times10^{-1}k^3+0.44637476k^4+0.40425797k^5 \\
& +0.17512580k^6+0.42260742\times10^{-1}k^7+0.57834201\times10^{-2}k^8+0.41852678\times10^{-3}k^9 \\
& +0.12400793\times10^{-4}k^{10} \\
& +(0.18236606+0.64614603k+0.11708272\times10k^2+0.10231753\times10k^3+0.36897983k^4 \\
& +0.90220141\times10^{-2}k^5-0.30064560\times10^{-1}k^6-0.84278893\times10^{-2}k^7 \\
& -0.92695932\times10^{-3}k^8-0.37202380\times10^{-4}k^9)n \\
& +(-0.44968668-0.92096825k-0.13046961\times10k^2-0.10184443\times10k^3-0.38944540k^4 \\
& -0.67910427\times10^{-1}k^5-0.31494141\times10^{-2}k^6+0.43712797\times10^{-3}k^7 \\
& +0.40302579\times10^{-4}k^8)n^2 \\
& +(-0.60522515\times10^{-1}-0.15496225k+0.13251630k^2+0.20675557k^3 \\
& +0.79060872\times10^{-1}k^4+0.11740451\times10^{-1}k^5+0.52083333\times10^{-3}k^6 \\
& -0.12400793\times10^{-4}k^7)n^3 \\
& +(0.27282298+0.34439196k+0.10619591k^2-0.46666463\times10^{-2}k^3 \\
& -0.59305826\times10^{-2}k^4-0.66460503\times10^{-3}k^5-0.10850694\times10^{-4}k^6)n^4 \\
& +(-0.11405037-0.10231933k-0.26261562\times10^{-1}k^2-0.13014051\times10^{-2}k^3 \\
& +0.20345052\times10^{-3}k^4+0.10850694\times10^{-4}k^5)n^5 \\
& +(0.20088619\times10^{-1}+0.12467956\times10^{-1}k+0.21255492\times10^{-2}k^2+0.69173177\times10^{-4}k^3 \\
& -0.27126735\times10^{-5}k^4)n^6 \\
& +(-0.17647637\times10^{-2}-0.70224337\times10^{-3}k-0.65491690\times10^{-4}k^2 \\
& -0.77504960\times10^{-6}k^3)n^7 \\
& +(0.78643314\times10^{-4}+0.17329624\times10^{-4}k+0.62972780\times10^{-6}k^2)n^8 \\
& +(-0.16348702\times10^{-5}-0.14532179\times10^{-6}k)n^9+0.12110150\times10^{-7}n^{10}
\end{aligned}
$$

$$
\begin{aligned}
\Xi_{7,3}^{n,k} = \; & -0.96550521\times10^{-2}k+0.76694256\times10^{-1}k^2+0.22972392k^3+0.26727727k^4 \\
& +0.11631402k^5-0.55582701\times10^{-2}k^6-0.19683159\times10^{-1}k^7-0.60391864\times10^{-2}k^8 \\
& -0.74404761\times10^{-3}k^9-0.33068783\times10^{-4}k^{10} \\
& +(0.63481737\times10^{-2}+0.67038427\times10^{-1}k+0.20346539k^2+0.24878521\times10^{-1}k^3 \\
& -0.28906091k^4-0.22798755k^5-0.60742187\times10^{-1}k^6-0.41211970\times10^{-2}k^7 \\
& +0.57043650\times10^{-3}k^8+0.66137566\times10^{-4}k^9)n \\
& +(-0.18833421-0.32821450k-0.58475452k^2-0.39138365k^3-0.27601582\times10^{-3}k^4 \\
& +0.66861978\times10^{-1}k^5+0.17657696\times10^{-1}k^6+0.11656746\times10^{-2}k^7 \\
& -0.24801587\times10^{-4}k^8)n^2 \\
& +(0.11741485+0.20775560k+0.46822148k^2+0.31561144k^3+0.59642650\times10^{-1}k^4 \\
& -0.34143518\times10^{-2}k^5-0.13888888\times10^{-2}k^6-0.33068783\times10^{-4}k^7)n^3 \\
& +(0.12197782+0.11183087k-0.80546060\times10^{-1}k^2-0.67100242\times10^{-1}k^3 \\
& -0.10751230\times10^{-1}k^4+0.28935185\times10^{-4}k^6)n^4 \\
& +(-0.11711448-0.88845937\times10^{-1}k-0.35115559\times10^{-2}k^2+0.54235387\times10^{-2}k^3 \\
& +0.54253472\times10^{-3}k^4)n^5 \\
& +(0.32379150\times10^{-1}+0.16669831\times10^{-1}k+0.10520652\times10^{-2}k^2-0.18446180\times10^{-3}k^3 \\
& -0.72337963\times10^{-5}k^4)n^6 \\
& +(-0.37837275\times10^{-2}-0.11597325\times10^{-2}k-0.40302579\times10^{-4}k^2 \\
& +0.20667989\times10^{-5}k^3)n^7 \\
& +(0.19744388\times10^{-3}+0.30808222\times10^{-4}k+0.38752480\times10^{-6}k^2)n^8 \\
& +(-0.43596540\times10^{-5}-0.25834987\times10^{-6}k)n^9+0.32293733\times10^{-7}n^{10}
\end{aligned}
$$

$$\Xi_{6,4}^{n,k} = 0.65140335\times10^{-2}k + 0.24194336\times10^{-1}k^2 + 0.46891954\times10^{-1}k^3 + 0.88596144\times10^{-1}k^4$$
$$+0.14601474\times10^{-1}k^5 - 0.32910834\times10^{-1}k^6 - 0.11886936\times10^{-1}k^7$$
$$+0.71614584\times10^{-3}k^8 + 0.65104166\times10^{-3}k^9 + 0.57870370\times10^{-4}k^{10}$$
$$+(-0.47019776\times10^{-4} - 0.20221174\times10^{-1}k + 0.94976352\times10^{-1}k^2$$
$$-0.35309980\times10^{-2}k^3 - 0.24917664k^4 - 0.10212243k^5 + 0.23920356\times10^{-1}k^6$$
$$+0.13635705\times10^{-1}k^7 + 0.99826388\times10^{-3}k^8 - 0.57870370\times10^{-4}k^9)n$$
$$+(-0.70167822\times10^{-1} - 0.27030207\times10^{-1}k - 0.28737996k^2 - 0.13070667k^3$$
$$+0.18077465k^4 + 0.83542208\times10^{-1}k^5 + 0.24775749\times10^{-3}k^6 - 0.20399305\times10^{-2}k^7$$
$$-0.43402777\times10^{-4}k^8)n^2$$
$$+(0.73226136\times10^{-1} + 0.71710881\times10^{-1}k + 0.33809622k^2 + 0.14453373k^3$$
$$-0.50681785\times10^{-1}k^4 - 0.20066550\times10^{-1}k^5 - 0.34722222\times10^{-3}k^6$$
$$+0.57870370\times10^{-4}k^7)n^3$$
$$+(0.58041268\times10^{-1} + 0.11329990\times10^{-1}k - 0.16910818k^2 - 0.61817875\times10^{-1}k^3$$
$$+0.39537217\times10^{-2}k^4 + 0.13292100\times10^{-2}k^5 + 0.72337963\times10^{-5}k^6)n^4$$
$$+(-0.94525767\times10^{-1} - 0.40210412\times10^{-1}k + 0.31115383\times10^{-1}k^2$$
$$+0.80263489\times10^{-2}k^3 - 0.13563368\times10^{-3}k^4 - 0.21701388\times10^{-4}k^5)n^5$$
$$+(0.38030836\times10^{-1} + 0.11802277\times10^{-1}k - 0.23402461\times10^{-2}k^2 - 0.32280815\times10^{-3}k^3$$
$$+0.18084490\times10^{-5}k^4)n^6$$
$$+(-0.56784170\times10^{-2} - 0.96819842\times10^{-3}k + 0.70529513\times10^{-4}k^2$$
$$+0.36168981\times10^{-5}k^3)n^7$$
$$+(0.33264159\times10^{-3} + 0.26957194\times10^{-4}k - 0.67816839\times10^{-6}k^2)n^8$$
$$+(-0.76293945\times10^{-5} - 0.22605613\times10^{-6}k)n^9 + 0.56514033\times10^{-7}n^{10}$$

$$\Xi_{5,5}^{n,k} = -0.17112629\times10^{-1}k^2 + 0.75193954\times10^{-1}k^4 - 0.31827528\times10^{-1}k^6$$
$$+0.30815972\times10^{-2}k^8 - 0.69444444\times10^{-4}k^{10}$$
$$+(-0.22460938\times10^{-1} + 0.17841661k^2 - 0.24329664k^4 + 0.50314669\times10^{-1}k^6$$
$$-0.19965278\times10^{-2}k^8)n$$
$$+(0.33593751\times10^{-2} - 0.34987521k^2 + 0.23698086k^4 - 0.20930989\times10^{-1}k^6$$
$$+0.86805554\times10^{-4}k^8)n^2$$
$$+(0.19703504\times10^{-1} + 0.32889879k^2 - 0.95540364\times10^{-1}k^4 + 0.20833333\times10^{-2}k^6)n^3$$
$$+(0.51089138\times10^{-1} - 0.17411363k^2 + 0.16126844\times10^{-1}k^4 - 0.43402777\times10^{-4}k^6)n^4$$
$$+(-0.84541489\times10^{-1} + 0.46090358\times10^{-1}k^2 - 0.81380208\times10^{-3}k^4)n^5$$
$$+(0.38907402\times10^{-1} - 0.43477376\times10^{-2}k^2 + 0.10850694\times10^{-4}k^4)n^6$$
$$+(-0.64419216\times10^{-2} + 0.14105903\times10^{-3}k^2)n^7$$
$$+(0.38045247\times10^{-3} - 0.13563368\times10^{-5}k^2)n^8 - 0.91552733\times10^{-5}n^9$$
$$+0.67816840\times10^{-7}n$$

4.5 Generalized Newcomb operators for $\varkappa = 1$

$$\Xi_{0,0}^{n,k} = 0.09999999\text{x}10$$

$$\Xi_{1,0}^{n,k} = 0.09999999\text{x}10k-0.50000000n$$

$$\Xi_{2,0}^{n,k} = 0.62500000k+0.50000000k^2+(-0.37500000-0.50000000k)n+0.12500000n^2$$

$$\Xi_{1,1}^{n,k} = -0.09999999\text{x}10k^2-0.17500000\text{x}10n+0.25000000n^2$$

$$\begin{aligned}\Xi_{3,0}^{n,k} = {} & 0.54166666k+0.62500000k^2+0.16666666k^3 \\ & +(-0.35416666-0.68750000k-0.25000000k^2)n+(0.18750000+0.12500000k)n^2 \\ & -0.20833333\text{x}10^{-1}n^3\end{aligned}$$

$$\begin{aligned}\Xi_{2,1}^{n,k} = {} & 0.25000000k-0.62500000k^2-0.50000000k^3+(-0.16875000\text{x}10k+0.25000000k^2)n \\ & +(0.10625000\text{x}10+0.12500000k)n^2-0.62500000\text{x}10^{-1}n^3\end{aligned}$$

$$\begin{aligned}\Xi_{4,0}^{n,k} = {} & 0.53645833k+0.73697916k^2+0.31250000k^3+0.41666666\text{x}10^{-1}k^4 \\ & +(-0.36979166-0.85937500k-0.50000000k^2-0.83333333\text{x}10^{-1}k^3)n \\ & +(0.24739583+0.26562500k+0.62500000\text{x}10^{-1}k^2)n^2 \\ & +(-0.46875000\text{x}10^{-1}-0.20833333\text{x}10^{-1}k)n^3+0.26041666\text{x}10^{-2}n^4\end{aligned}$$

$$\begin{aligned}\Xi_{3,1}^{n,k} = {} & 0.23958333k-0.29166666k^2-0.62500000k^3-0.16666666k^4 \\ & +(-0.52083333\text{x}10^{-1}-0.11354166\text{x}10k-0.50000000k^2+0.16666666k^3)n \\ & +(0.83333333+0.10312500\text{x}10k)n^2+(-0.31250000-0.41666666\text{x}10^{-1}k)n^3 \\ & +0.10416666\text{x}10^{-1}n^4\end{aligned}$$

$$\begin{aligned}\Xi_{2,2}^{n,k} = {} & -0.89062500k^2+0.25000000k^4+(-0.78125000+0.20000000\text{x}10k^2)n \\ & +(0.16718750\text{x}10-0.12500000k^2)n^2-0.53125000n^3+0.15625000\text{x}10^{-1}n^4\end{aligned}$$

$$\begin{aligned}\Xi_{5,0}^{n,k} = {} & 0.57135417k+0.87500000k^2+0.46614583k^3+0.10416666k^4+0.83333333\text{x}10^{-2}k^5 \\ & +(-0.40859374-0.10625000\text{x}10k-0.77994791k^2-0.21875000k^3 \\ & -0.20833333\text{x}10^{-1}k^4)n \\ & +(0.31770833+0.43229166k+0.17187500k^2+0.20833333\text{x}10^{-1}k^3)n^2 \\ & +(-0.79427083\text{x}10^{-1}-0.59895833\text{x}10^{-1}k-0.10416666\text{x}10^{-1}k^2)n^3 \\ & +(0.78125000\text{x}10^{-2}+0.26041666\text{x}10^{-2}k)n^4-0.26041666\text{x}10^{-3}n^5\end{aligned}$$

$$\begin{aligned}\Xi_{4,1}^{n,k} = {} & 0.27343749k-0.14062500k^2-0.61197916k^3-0.31250000k^4-0.41666666\text{x}10^{-1}k^5 \\ & +(-0.97656248\text{x}10^{-1}-0.11119791\text{x}10k-0.72786458k^2+0.52083331\text{x}10^{-1}k^3 \\ & +0.62500000\text{x}10^{-1}k^4)n \\ & +(0.83072916+0.14166666\text{x}10k+0.42187500k^2-0.20833333\text{x}10^{-1}k^3)n^2 \\ & +(-0.45182291-0.30468750k-0.10416666\text{x}10^{-1}k^2)n^3 \\ & +(0.59895833\text{x}10^{-1}+0.78125000\text{x}10^{-2}k)n^4-0.13020833\text{x}10^{-2}n^5\end{aligned}$$

$$\begin{aligned}\Xi_{3,2}^{n,k} = {} & 0.65104166\text{x}10^{-1}k-0.73437500k^2-0.49479166k^3+0.20833333k^4 \\ & +0.83333333\text{x}10^{-1}k^5 \\ & +(0.12890625-0.11744791\text{x}10k+0.15078124\text{x}10k^2+0.93750000k^3 \\ & -0.41666666\text{x}10^{-1}k^4)n \\ & +(0.54947916+0.14843749\text{x}10k-0.59375000k^2-0.41666666\text{x}10^{-1}k^3)n^2 \\ & +(-0.12083333\text{x}10-0.24479166k+0.20833333\text{x}10^{-1}k^2)n^3 \\ & +(0.14062500+0.52083333\text{x}10^{-2}k)n^4-0.26041666\text{x}10^{-2}n^5\end{aligned}$$

$$
\begin{aligned}
\Xi^{n,k}_{6,0} = \; & 0.63697916k+0.10533419x10k^{2}+0.64746094k^{3}+0.18793403k^{4} \\
& +0.26041666x10^{-1}k^{5}+0.13888888x10^{-2}k^{6} \\
& +(-0.46822916-0.13184027x10k-0.11201172x10k^{2}-0.40928819k^{3} \\
& -0.67708333x10^{-1}k^{4}-0.41666666x10^{-2}k^{5})n \\
& +(0.40568576+0.64095052k+0.33300781k^{2}+0.70312500x10^{-1}k^{3} \\
& +0.52083333x10^{-2}k^{4})n^{2} \\
& +(-0.12141927-0.12000868k-0.36458333x10^{-1}k^{2}-0.34722222x10^{-2}k^{3})n^{3} \\
& +(0.16167534x10^{-1}+0.94401041x10^{-2}k+0.13020833x10^{-2}k^{2})n^{4} \\
& +(-0.97656250x10^{-3}-0.26041666x10^{-3}k)n^{5}+0.21701388x10^{-4}n^{6}
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{5,1} = \; & 0.33489583k-0.12760421x10^{-1}k^{2}-0.59895833k^{3}-0.42447916k^{4}-0.10416666k^{5} \\
& -0.83333333x10^{-2}k^{6} \\
& +(-0.15156250-0.12611979x10k-0.98242188k^{2}-0.62499997x10^{-1}k^{3} \\
& +0.93749999x10^{-1}k^{4}+0.16666666x10^{-1}k^{5})n \\
& +(0.91979166+0.18203124x10k+0.86393229k^{2}+0.83333333x10^{-1}k^{3} \\
& -0.10416666x10^{-1}k^{4})n^{2} \\
& +(-0.59830729-0.60677083k-0.13541666k^{2})n^{3} \\
& +(0.12174479+0.58593750x10^{-1}k+0.26041666x10^{-2}k^{2})n^{4} \\
& +(-0.84635416x10^{-2}-0.10416666x10^{-2}k)n^{5}+0.13020833x10^{-3}n^{6}
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{4,2} = \; & 0.13020832x10^{-1}k-0.64778645k^{2}-0.74446614k^{3}+0.65104160x10^{-2}k^{4} \\
& +0.13020833k^{5}+0.20833333x10^{-1}k^{6} \\
& +(0.13281250-0.73177084k+0.70345052k^{2}+0.13528645x10k^{3}+0.23437500k^{4} \\
& -0.20833333x10^{-1}k^{5})n \\
& +(0.50716145+0.15302734x10k-0.83007813x10^{-1}k^{2}-0.37760416k^{3} \\
& -0.52083333x10^{-2}k^{4})n^{2} \\
& +(-0.11337890x10-0.11035156x10k+0.36458333x10^{-1}k^{2}+0.10416666x10^{-1}k^{3})n^{3} \\
& +(0.40397135+0.88867188x10^{-1}k-0.13020833x10^{-2}k^{2})n^{4} \\
& +(-0.25065104x10^{-1}-0.13020833x10^{-2}k)n^{5}+0.32552083x10^{-3}n^{6}
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{3,3} = \; & -0.78559028k^{2}+0.46006944k^{4}-0.27777778x10^{-1}k^{6} \\
& +(-0.48437500+0.27981770x10k^{2}-0.52083333k^{4})n \\
& +(0.14027777x10-0.22278645x10k^{2}+0.20833333x10^{-1}k^{4})n^{2} \\
& +(-0.14804687x10+0.27083333k^{2})n^{3}+(0.59678819-0.52083333x10^{-2}k^{2})n^{4} \\
& -0.35156250x10^{-1}n^{5}+0.43402778x10^{-3}n^{6}
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{7,0} = \; & 0.73277839k+0.12846570x10k^{2}+0.87345920k^{3}+0.29937065k^{4} \\
& +0.55121527x10^{-1}k^{5}+0.52083333x10^{-2}k^{6}+0.19841269x10^{-3}k^{7} \\
& +(-0.55041077-0.16466471x10k-0.15512261x10k^{2}-0.67084418k^{3}-0.14778645k^{4} \\
& -0.16145833x10^{-1}k^{5}-0.69444443x10^{-3}k^{6})n \\
& +(0.51830512+0.91026475k+0.56103516k^{2}+0.15809461k^{3}+0.20833333x10^{-1}k^{4} \\
& +0.10416666x10^{-2}k^{5})n^{2} \\
& +(-0.17667101-0.20762803k-0.84364148x10^{-1}k^{2}-0.14322916x10^{-1}k^{3} \\
& -0.86805555x10^{-3}k^{4})n^{3} \\
& +(0.28700086x10^{-1}+0.22460938x10^{-1}k+0.55338541x10^{-2}k^{2} \\
& +0.43402778x10^{-3}k^{3})n^{4} \\
& +(-0.23871527x10^{-2}-0.11393229x10^{-2}k-0.13020833x10^{-3}k^{2})n^{5} \\
& +(0.97656249x10^{-4}+0.21701388x10^{-4}k)n^{6}-0.15500992x10^{-5}n^{7}
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{6,1} = {} & 0.42491319k+0.13270400k^{2}-0.58263887k^{3}-0.52940538k^{4}-0.17751735k^{5}\\
& -0.26041666\times10^{-1}k^{6}-0.13888888\times10^{-2}k^{7}\\
& +(-0.22146267-0.15382270\times10k-0.13371744\times10k^{2}-0.21294488k^{3}+0.11219618k^{4}\\
& +0.40104166\times10^{-1}k^{5}+0.34722222\times10^{-2}k^{6})n\\
& +(0.10800021\times10+0.23384766\times10k+0.13933919\times10k^{2}+0.27007379k^{3}\\
& +0.77610213\times10^{-10}k^{4}-0.31249999\times10^{-2}k^{5})n^{2}\\
& +(-0.78080512-0.97878689k-0.35248481k^{2}-0.35156250\times10^{-1}k^{3}\\
& +0.86805555\times10^{-3}k^{4})n^{3}\\
& +(0.20079210+0.14930555k+0.27018229\times10^{-1}k^{2}+0.43402778\times10^{-3}k^{3})n^{4}\\
& +(-0.21755642\times10^{-1}-0.83007813\times10^{-2}k-0.39062499\times10^{-3}k^{2})n^{5}\\
& +(0.94401041\times10^{-3}+0.10850694\times10^{-3}k)n^{6}-0.10850694\times10^{-4}n^{7}
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{5,2} = {} & -0.16601565\times10^{-1}k-0.71250000k^{2}-0.94023438k^{3}-0.20800781k^{4}+0.11588541k^{5}\\
& +0.46875000\times10^{-1}k^{6}+0.41666666\times10^{-2}k^{7}\\
& +(0.15107422-0.61458333k+0.57226563k^{2}+0.14643554\times10k^{3}+0.55013020k^{4}\\
& +0.25520833\times10^{-1}k^{5}-0.62499999\times10^{-2}k^{6})n\\
& +(0.56184895+0.17567708\times10k+0.26090495k^{2}-0.50292968k^{3}-0.12500000k^{4}\\
& +0.10416666\times10^{-2}k^{5})n^{2}\\
& +(-0.12462239\times10-0.16437174\times10k-0.30875651k^{2}+0.63802083\times10^{-1}k^{3}\\
& +0.26041666\times10^{-2}k^{4})n^{3}\\
& +(0.59261068+0.37565104k+0.15950520\times10^{-1}k^{2}-0.13020833\times10^{-2}k^{3})n^{4}\\
& +(-0.85774739\times10^{-1}-0.18066406\times10^{-1}k-0.13020833\times10^{-3}k^{2})n^{5}\\
& +(0.33528645\times10^{-2}+0.19531249\times10^{-3}k)n^{6}-0.32552083\times10^{-4}n^{7}
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{4,3} = {} & -0.20616324\times10^{-2}k-0.70486110k^{2}-0.38031684k^{3}+0.43804253k^{4}+0.15407985k^{5}\\
& -0.26041666\times10^{-1}k^{6}-0.69444444\times10^{-2}k^{7}\\
& +(0.17290581-0.78949653k+0.23161349\times10k^{2}+0.13675673\times10k^{3}-0.51453993k^{4}\\
& -0.16406250k^{5}+0.34722222\times10^{-2}k^{6})n\\
& +(0.87890624\times10^{-1}+0.15993923\times10k-0.22153320\times10k^{2}-0.94824219k^{3}\\
& +0.10416666k^{4}+0.52083333\times10^{-2}k^{5})n^{2}\\
& +(-0.12484266\times10-0.10392252\times10k+0.74560547k^{2}+0.84635416\times10^{-1}k^{3}\\
& -0.26041666\times10^{-2}k^{4})n^{3}\\
& +(0.10581055\times10+0.24880642k-0.48502604\times10^{-1}k^{2}-0.13020833\times10^{-2}k^{3})n^{4}\\
& +(-0.16525608-0.10904948\times10^{-1}k+0.65104166\times10^{-3}k^{2})n^{5}\\
& +(0.60221354\times10^{-2}+0.10850694\times10^{-3}k)n^{6}-0.54253472\times10^{-4}n^{7}
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{8,0} = {} & 0.86247984k+0.15842676\times10k^{2}+0.11626329\times10k^{3}+0.44837172k^{4}\\
& +0.99121094\times10^{-1}k^{5}+0.12651909\times10^{-1}k^{6}+0.86805555\times10^{-3}k^{7}\\
& +0.24801587\times10^{-4}k^{8}\\
& +(-0.65930524-0.20703644\times10k-0.21092420\times10k^{2}-0.10282796\times10k^{3}\\
& -0.27246094k^{4}-0.40277778\times10^{-1}k^{5}-0.31249999\times10^{-2}k^{6}-0.99206349\times10^{-4}k^{7})n\\
& +(0.66387454+0.12634195\times10k+0.87946641k^{2}+0.29859754k^{3}\\
& +0.53331163\times10^{-1}k^{4}+0.48177082\times10^{-2}k^{5}+0.17361110\times10^{-3}k^{6})n^{2}\\
& +(-0.25014377-0.33263075k-0.16312663k^{2}-0.37597656\times10^{-1}k^{3}\\
& -0.41232638\times10^{-2}k^{4}-0.17361110\times10^{-3}k^{5})n^{3}\\
& +(0.46961127\times10^{-1}+0.44433594\times10^{-1}k+0.14885796\times10^{-1}k^{2}+0.21158854\times10^{-2}k^{3}\\
& +0.10850694\times10^{-3}k^{4})n^{4}\\
& +(-0.48285589\times10^{-2}-0.31385633\times10^{-2}k-0.65104166\times10^{-3}k^{2}\\
& -0.43402777\times10^{-4}k^{3})n^{5}\\
& +(0.27533636\times10^{-3}+0.11121961\times10^{-3}k+0.10850694\times10^{-4}k^{2})n^{6}\\
& +(-0.81380208\times10^{-5}-0.15500992\times10^{-5}k)n^{7}+0.96881200\times10^{-7}n^{8}
\end{aligned}
$$

$$
\begin{aligned}
\Xi_{7,1}^{n,k} = {} & 0.55005580k+0.32092168k^{2}-0.55099282k^{3}-0.63648003k^{4}-0.26334635k^{5} \\
& -0.53038194\times10^{-1}k^{6}-0.52083333\times10^{-2}k^{7}-0.19841269\times10^{-3}k^{8} \\
& +(-0.31501116-0.19533481\times10k-0.18411838\times10k^{2}-0.43425021k^{3}+0.11545138k^{4} \\
& +0.69444443\times10^{-1}k^{5}+0.11111110\times10^{-1}k^{6}+0.59523808\times10^{-3}k^{7})n \\
& +(0.13160203\times10+0.30317328\times10k+0.20819010\times10k^{2}+0.55292426k^{3} \\
& +0.39496528\times10^{-1}k^{4}-0.54687501\times10^{-2}k^{5}-0.69444443\times10^{-3}k^{6})n^{2} \\
& +(-0.10192437\times10-0.14629503\times10k-0.67887370k^{2}-0.12033420k^{3} \\
& -0.52083333\times10^{-2}k^{4}+0.34722222\times10^{-3}k^{5})n^{3} \\
& +(0.30529513+0.28879123k+0.84174262\times10^{-1}k^{2}+0.77039930\times10^{-2}k^{3})n^{4} \\
& +(-0.42778862\times10^{-1}-0.25428602\times10^{-1}k-0.39062500\times10^{-2}k^{2} \\
& -0.86805554\times10^{-4}k^{3})n^{5} \\
& +(0.29025607\times10^{-2}+0.92773437\times10^{-3}k+0.43402777\times10^{-4}k^{2})n^{6} \\
& +(-0.86805554\times10^{-4}-0.93005951\times10^{-5}k)n^{7}+0.77504960\times10^{-6}n^{8}
\end{aligned}
$$

$$
\begin{aligned}
\Xi_{6,2}^{n,k} = {} & -0.39865454\times10^{-1}k-0.84200305k^{2}-0.12016547\times10k^{3}-0.43875597k^{4} \\
& +0.62174477\times10^{-1}k^{5}+0.65190971\times10^{-1}k^{6}+0.12152778\times10^{-1}k^{7} \\
& +0.69444443\times10^{-3}k^{8} \\
& +(0.17805989-0.59287108k+0.56983508k^{2}+0.16756564\times10k^{3}+0.84852431k^{4} \\
& +0.12500000k^{5}-0.41666666\times10^{-2}k^{6}-0.13888888\times10^{-2}k^{7})n \\
& +(0.67081163+0.21424750\times10k+0.62111003k^{2}-0.57288954k^{3}-0.27148437k^{4} \\
& -0.26302083\times10^{-1}k^{5}+0.69444443\times10^{-3}k^{6})n^{2} \\
& +(-0.14719835\times10-0.22597819\times10k-0.74169923k^{2}+0.33528645\times10^{-1}k^{3} \\
& +0.26909722\times10^{-1}k^{4}+0.34722222\times10^{-3}k^{5})n^{3} \\
& +(0.80937771+0.73426649k+0.14005534k^{2}-0.42317708\times10^{-2}k^{3} \\
& -0.43402778\times10^{-3}k^{4})n^{4} \\
& +(-0.16509331-0.80745442\times10^{-1}k-0.52083333\times10^{-2}k^{2}+0.86805554\times10^{-4}k^{3})n^{5} \\
& +(0.13210720\times10^{-1}+0.25987413\times10^{-2}k+0.43402777\times10^{-4}k^{2})n^{6} \\
& +(-0.35807291\times10^{-3}-0.21701388\times10^{-4}k)n^{7}+0.27126735\times10^{-5}n^{8}
\end{aligned}
$$

$$
\begin{aligned}
\Xi_{5,3}^{n,k} = {} & -0.98480903\times10^{-1}k-0.72012804k^{2}-0.59424371k^{3}+0.30939670k^{4}+0.26920573k^{5} \\
& +0.14149305\times10^{-1}k^{6}-0.12152778\times10^{-1}k^{7}-0.13888888\times10^{-2}k^{8} \\
& +(0.21861979-0.28324652k+0.17904893\times10k^{2}+0.20645562\times10k^{3} \\
& -0.73784721\times10^{-1}k^{4}-0.29722222k^{5}-0.33333333\times10^{-1}k^{6}+0.13888888\times10^{-2}k^{7})n \\
& +(0.30707454\times10^{-2}+0.11712077\times10k-0.15194010\times10k^{2}-0.17073838\times10k^{3} \\
& -0.12282986k^{4}+0.49739583\times10^{-1}k^{5}+0.69444443\times10^{-3}k^{6})n^{2} \\
& +(-0.11717284\times10-0.16695692\times10k+0.59554036k^{2}+0.44433594k^{3} \\
& +0.52083333\times10^{-2}k^{4}-0.10416666\times10^{-2}k^{5})n^{3} \\
& +(0.12151910\times10+0.86322699k-0.84174262\times10^{-1}k^{2}-0.23111979\times10^{-1}k^{3})n^{4} \\
& +(-0.37345377-0.97759331\times10^{-1}k+0.39062500\times10^{-2}k^{2}+0.26041666\times10^{-3}k^{3})n^{5} \\
& +(0.31125217\times10^{-1}+0.28591579\times10^{-2}k-0.43402777\times10^{-4}k^{2})n^{6} \\
& +(-0.78124999\times10^{-3}-0.21701388\times10^{-4}k)n^{7}+0.54253472\times10^{-5}n^{8}
\end{aligned}
$$

$$
\begin{aligned}
\Xi_{4,4}^{n,k} = {} & -0.68611653k^{2}+0.63493517k^{4}-0.77907985\times10^{-1}k^{6}+0.17361110\times10^{-2}k^{8} \\
& +(-0.36035156+0.31802028\times10k^{2}-0.12354600\times10k^{4}+0.59027778\times10^{-1}k^{6})n \\
& +(0.10889485\times10-0.41261528\times10k^{2}+0.60297309k^{4}-0.17361110\times10^{-2}k^{6})n^{2} \\
& +(-0.16460232\times10+0.19763183\times10k^{2}-0.45572916\times10^{-1}k^{4})n^{3} \\
& +(0.13699611\times10-0.30988227k^{2}+0.65104166\times10^{-3}k^{4})n^{4} \\
& +(-0.49262153+0.11718750\times10^{-1}k^{2})n^{5} \\
& +(0.41083441\times10^{-1}-0.10850694\times10^{-3}k^{2})n^{6}-0.10036892\times10^{-2}n^{7} \\
& +0.67816840\times10^{-5}n^{8}
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{9,0} = {} & 0.10329466\times10k+0.19720044\times10k^{2}+0.15375723\times10k^{3}+0.64850803k^{4} \\
& +0.16339269k^{5}+0.25358073\times10^{-1}k^{6}+0.23799189\times10^{-2}k^{7}+0.12400793\times10^{-3}k^{8} \\
& +0.27557319\times10^{-5}k^{9} \\
& +(-0.80125679-0.26190402\times10k-0.28394803\times10k^{2}-0.15162732\times10k^{3} \\
& -0.45855453k^{4}-0.82535807\times10^{-1}k^{5}-0.87709779\times10^{-2}k^{6}-0.50843254\times10^{-3}k^{7} \\
& -0.12400793\times10^{-4}k^{8})n \\
& +(0.85298239+0.17303805\times10k+0.13214056\times10k^{2}+0.51280200k^{3}+0.11166721k^{4} \\
& +0.13834635\times10^{-1}k^{5}+0.91145833\times10^{-3}k^{6}+0.24801587\times10^{-4}k^{7})n^{2} \\
& +(-0.34838222-0.50899386k-0.28571890k^{2}-0.80396863\times10^{-1}k^{3} \\
& -0.12107566\times10^{-1}k^{4}-0.93315972\times10^{-3}k^{5}-0.28935185\times10^{-4}k^{6})n^{3} \\
& +(0.73151313\times10^{-1}+0.79335077\times10^{-1}k+0.32491048\times10^{-1}k^{2}+0.63499168\times10^{-2}k^{3} \\
& +0.59678818\times10^{-3}k^{4}+0.21701388\times10^{-4}k^{5})n^{4} \\
& +(-0.87844282\times10^{-2}-0.69892035\times10^{-2}k-0.19958495\times10^{-2}k^{2} \\
& -0.24414063\times10^{-3}k^{3}-0.10850694\times10^{-4}k^{4})n^{5} \\
& +(0.62527126\times10^{-3}+0.34812644\times10^{-3}k+0.62391493\times10^{-4}k^{2} \\
& +0.36168981\times10^{-5}k^{3})n^{6} \\
& +(-0.25996455\times10^{-4}-0.91068327\times10^{-5}k-0.77504960\times10^{-6}k^{2})n^{7} \\
& +(0.58128719\times10^{-6}+0.96881200\times10^{-7}k)n^{8}-0.53822888\times10^{-8}n^{9}
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{8,1} = {} & 0.72096694k+0.57736816k^{2}-0.48801250k^{3}-0.74782172k^{4}-0.36503838k^{5} \\
& -0.90614148\times10^{-1}k^{6}-0.12304687\times10^{-1}k^{7}-0.86805555\times10^{-3}k^{8} \\
& -0.24801587\times10^{-4}k^{9} \\
& +(-0.44171651-0.25410088\times10k-0.25547615\times10k^{2}-0.76801895k^{3} \\
& +0.93183386\times10^{-1}k^{4}+0.10408528k^{5}+0.23795572\times10^{-1}k^{6}+0.23437499\times10^{-2}k^{7} \\
& +0.86805554\times10^{-4}k^{8})n \\
& +(0.16444973\times10+0.39682260\times10k+0.30114894\times10k^{2}+0.96970215k^{3} \\
& +0.11991373k^{4}-0.36349825\times10^{-2}k^{5}-0.20399305\times10^{-2}k^{6}-0.12400793\times10^{-3}k^{7})n^{2} \\
& +(-0.13357791\times10-0.21111084\times10k-0.11540181\times10k^{2}-0.27553304k^{3} \\
& -0.26394314\times10^{-1}k^{4}-0.10850694\times10^{-3}k^{5}+0.86805554\times10^{-4}k^{6})n^{3} \\
& +(0.44611816+0.49448310k+0.18829346k^{2}+0.29303657\times10^{-1}k^{3} \\
& +0.14648438\times10^{-2}k^{4}-0.21701388\times10^{-4}k^{5})n^{4} \\
& +(-0.74366929\times10^{-1}-0.57137044\times10^{-1}k-0.14053684\times10^{-1}k^{2} \\
& -0.11664496\times10^{-2}k^{3}-0.10850694\times10^{-4}k^{4})n^{5} \\
& +(0.66053602\times10^{-2}+0.32931858\times10^{-2}k+0.44216579\times10^{-3}k^{2} \\
& +0.10850694\times10^{-4}k^{3})n^{6} \\
& +(-0.30856661\times10^{-3}-0.85449218\times10^{-4}k-0.38752480\times10^{-5}k^{2})n^{7} \\
& +(0.67816840\times10^{-5}+0.67816839\times10^{-6}k)n^{8}-0.48440599\times10^{-7}n^{9}
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{7,2} = {} & -0.60388957\times10^{-1}k-0.10263148\times10k^{2}-0.15543179\times10k^{3}-0.72711860k^{4} \\
& -0.26418725\times10^{-1}k^{5}+0.72699651\times10^{-1}k^{6}+0.21874999\times10^{-1}k^{7} \\
& +0.24801587\times10^{-2}k^{8}+0.99206349\times10^{-4}k^{9} \\
& +(0.21267410-0.62819614k+0.61803044k^{2}+0.20004353\times10k^{3}+0.12048651\times10k^{4} \\
& +0.26584201k^{5}+0.12586805\times10^{-1}k^{6}-0.26785713\times10^{-2}k^{7}-0.24801587\times10^{-3}k^{8})n \\
& +(0.83538043+0.27090596\times10k+0.10812025\times10k^{2}-0.61513399k^{3}-0.44189453k^{4} \\
& -0.81770832\times10^{-1}k^{5}-0.34722222\times10^{-2}k^{6}+0.19841269\times10^{-3}k^{7})n^{2} \\
& +(-0.18114275\times10-0.30640842\times10k-0.13154419\times10k^{2}-0.66460506\times10^{-1}k^{3} \\
& +0.52300347\times10^{-1}k^{4}+0.68576389\times10^{-2}k^{5})n^{3} \\
& +(0.10892713\times10+0.12034735\times10k+0.36565484k^{2}+0.20670573\times10^{-1}k^{3} \\
& -0.32552083\times10^{-2}k^{4}-0.86805554\times10^{-4}k^{5})n^{4} \\
& +(-0.27061564-0.19158936k-0.33398438\times10^{-1}k^{2}-0.43402778\times10^{-3}k^{3} \\
& +0.43402777\times10^{-4}k^{4})n^{5} \\
& +(0.30863444\times10^{-1}+0.12546115\times10^{-1}k+0.88975694\times10^{-3}k^{2})n^{6} \\
& +(-0.15882704\times10^{-2}-0.29064360\times10^{-3}k-0.62003968\times10^{-5}k^{2})n^{7} \\
& +(0.31777033\times10^{-4}+0.19376240\times10^{-5}k)n^{8}-0.19376240\times10^{-6}n^{9}
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{6,3} = {} & -0.19743246k-0.90718859k^{2}-0.80148067k^{3}+0.18320584k^{4}+0.34085015k^{5}\\
& +0.72005207\times10^{-1}k^{6}-0.85069444\times10^{-2}k^{7}-0.34722222\times10^{-2}k^{8}\\
& -0.23148148\times10^{-3}k^{9}\\
& +(0.29405721+0.38680024\times10^{-1}k+0.20263159\times10k^{2}+0.25312568\times10k^{3}\\
& +0.41184941k^{4}-0.31032986k^{5}-0.92824074\times10^{-1}k^{6}-0.41666666\times10^{-2}k^{7}\\
& +0.34722222\times10^{-3}k^{8})n\\
& +(-0.63937729\times10^{-1}+0.10086543\times10k-0.15281752\times10k^{2}-0.21866798\times10k^{3}\\
& -0.51047092k^{4}+0.44704860\times10^{-1}k^{5}+0.13541666\times10^{-1}k^{6})n^{2}\\
& +(-0.12757625\times10-0.21568386\times10k+0.30136492k^{2}+0.75054254k^{3}+0.12254050k^{4}\\
& -0.47743054\times10^{-2}k^{5}-0.23148148\times10^{-3}k^{6})n^{3}\\
& +(0.14630479\times10+0.14446854\times10k+0.95716687\times10^{-1}k^{2}-0.91977720\times10^{-1}k^{3}\\
& -0.49913194\times10^{-2}k^{4}+0.86805554\times10^{-4}k^{5})n^{4}\\
& +(-0.56983168-0.32267117k-0.92827691\times10^{-2}k^{2}+0.32552083\times10^{-2}k^{3}\\
& +0.43402777\times10^{-4}k^{4})n^{5}\\
& +(0.84072536\times10^{-1}+0.21360496\times10^{-1}k+0.13020833\times10^{-3}k^{2}\\
& -0.28935185\times10^{-4}k^{3})n^{6}\\
& +(-0.44013128\times10^{-2}-0.45030382\times10^{-3}k)n^{7}\\
& +(0.81380208\times10^{-4}+0.27126735\times10^{-5}k)n^{8}-0.45211226\times10^{-6}n^{9}
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{5,4} = {} & -0.17558458\times10^{-1}k-0.61586913k^{2}-0.24603136k^{3}+0.64322645k^{4}+0.19427761k^{5}\\
& -0.79448784\times10^{-1}k^{6}-0.19140624\times10^{-1}k^{7}+0.17361110\times10^{-2}k^{8}\\
& +0.34722222\times10^{-3}k^{9}\\
& +(0.17661234-0.57207845k+0.27498955\times10k^{2}+0.14266452\times10k^{3}\\
& -0.12513434\times10k^{4}-0.38399523k^{5}+0.65212673\times10^{-1}k^{6}+0.14062499\times10^{-1}k^{7}\\
& -0.17361110\times10^{-3}k^{8})n\\
& +(-0.26169976+0.14258952\times10k-0.38859225\times10k^{2}-0.17674235\times10k^{3}\\
& +0.72078451k^{4}+0.17567274k^{5}-0.89409721\times10^{-2}k^{6}-0.34722222\times10^{-3}k^{7})n^{2}\\
& +(-0.85057101-0.13545912\times10k+0.24538139\times10k^{2}+0.73297797k^{3}-0.13633897k^{4}\\
& -0.10807291\times10^{-1}k^{5}+0.17361110\times10^{-3}k^{6})n^{3}\\
& +(0.15843356\times10+0.69802653k-0.68215603k^{2}-0.89694554\times10^{-1}k^{3}\\
& +0.61848958\times10^{-2}k^{4}+0.13020833\times10^{-3}k^{5})n^{4}\\
& +(-0.81875882-0.18122965k+0.58730740\times10^{-1}k^{2}+0.27669270\times10^{-2}k^{3}\\
& -0.65104166\times10^{-4}k^{4})n^{5}\\
& +(0.13851047+0.11759440\times10^{-1}k-0.15245225\times10^{-2}k^{2}-0.21701388\times10^{-4}k^{3})n^{6}\\
& +(-0.71885850\times10^{-2}-0.23600260\times10^{-3}k+0.10850694\times10^{-4}k^{2})n^{7}\\
& +(0.12749565\times10^{-3}+0.13563368\times10^{-5}k)n^{8}-0.67816839\times10^{-6}n^{9}
\end{aligned}
$$

$$
\begin{aligned}
\Xi_{10,0}^{n,k} = {} & 0.12541313\times10k+0.24738538\times10k^{2}+0.20272344\times10k^{3}+0.91804938k^{4} \\
& +0.25555501k^{5}+0.45655404\times10^{-1}k^{6}+0.52616825\times10^{-2}k^{7}+0.37874090\times10^{-3}k^{8} \\
& +0.15500992\times10^{-4}k^{9}+0.27557319\times10^{-6}k^{10} \\
& +(-0.98501691-0.33310774\times10k-0.38008564\times10k^{2}-0.21820702\times10k^{3} \\
& -0.72998835k^{4}-0.15142390k^{5}-0.19781720\times10^{-1}k^{6}-0.15857515\times10^{-2}k^{7} \\
& -0.71304563\times10^{-4}k^{8}-0.13778659\times10^{-5}k^{9})n \\
& +(0.10994266\times10+0.23506019\times10k+0.19323537\times10k^{2}+0.83054125k^{3} \\
& +0.20867075k^{4}+0.31815592\times10^{-1}k^{5}+0.29016565\times10^{-2}k^{6}+0.14570932\times10^{-3}k^{7} \\
& +0.31001984\times10^{-5}k^{8})n^{2} \\
& +(-0.48009884-0.75606086k-0.47061253k^{2}-0.15296009k^{3}-0.28379086\times10^{-1}k^{4} \\
& -0.30309606\times10^{-2}k^{5}-0.17361110\times10^{-3}k^{6}-0.41335978\times10^{-5}k^{7})n^{3} \\
& +(0.11033359+0.13284361k+0.62912269\times10^{-1}k^{2}+0.15163562\times10^{-1}k^{3} \\
& +0.19768608\times10^{-2}k^{4}+0.13292100\times10^{-3}k^{5}+0.36168981\times10^{-5}k^{6})n^{4} \\
& +(-0.14948159\times10^{-1}-0.13767993\times10^{-1}k-0.48538207\times10^{-2}k^{2} \\
& -0.82442671\times10^{-3}k^{3}-0.67816839\times10^{-4}k^{4}-0.21701389\times10^{-5}k^{5})n^{5} \\
& +(0.12526505\times10^{-2}+0.86189552\times10^{-3}k+0.21469681\times10^{-3}k^{2}+0.23057725\times10^{-4}k^{3} \\
& +0.90422453\times10^{-6}k^{4})n^{6} \\
& +(-0.65499765\times10^{-4}-0.31922355\times10^{-4}k-0.50378224\times10^{-5}k^{2} \\
& -0.25834987\times10^{-6}k^{3})n^{7} \\
& +(0.20748723\times10^{-5}+0.64183795\times10^{-6}k+0.48440599\times10^{-7}k^{2})n^{8} \\
& +(-0.36330450\times10^{-7}-0.53822888\times10^{-8}k)n^{9}+0.26911444\times10^{-9}n^{10}
\end{aligned}
$$

$$
\begin{aligned}
\Xi_{9,1}^{n,k} = {} & 0.95277967k+0.93365797k^{2}-0.37180925k^{3}-0.86115111k^{4}-0.48620243k^{5} \\
& -0.14109451k^{6}-0.23723234\times10^{-1}k^{7}-0.23303158\times10^{-2}k^{8}-0.12400793\times10^{-3}k^{9} \\
& -0.27557319\times10^{-5}k^{10} \\
& +(-0.61415533-0.33559866\times10k-0.35603725\times10k^{2}-0.12705424\times10k^{3} \\
& +0.27600823\times10^{-1}k^{4}+0.14236110k^{5}+0.42608868\times10^{-1}k^{6}+0.58883101\times10^{-2}k^{7} \\
& +0.40302579\times10^{-3}k^{8}+0.11022928\times10^{-4}k^{9})n \\
& +(0.20921699\times10+0.52370374\times10k+0.42875892\times10k^{2}+0.15794998\times10k^{3} \\
& +0.26090885k^{4}+0.67599832\times10^{-2}k^{5}-0.37199797\times10^{-2}k^{6}-0.48363095\times10^{-3}k^{7} \\
& -0.18601190\times10^{-4}k^{8})n^{2} \\
& +(-0.17588238\times10-0.29908594\times10k-0.18383905\times10k^{2}-0.53159858k^{3} \\
& -0.73450159\times10^{-1}k^{4}-0.34577546\times10^{-2}k^{5}+0.17361110\times10^{-3}k^{6} \\
& +0.16534391\times10^{-4}k^{7})n^{3} \\
& +(0.63774148+0.79276349k+0.36050110k^{2}+0.75379548\times10^{-1}k^{3} \\
& +0.70280851\times10^{-2}k^{4}+0.17361110\times10^{-3}k^{5}-0.72337963\times10^{-5}k^{6})n^{4} \\
& +(-0.12075732-0.11028985k-0.35721503\times10^{-1}k^{2}-0.49144603\times10^{-2}k^{3} \\
& -0.24414063\times10^{-3}k^{4})n^{5} \\
& +(0.12918486\times10^{-1}+0.84049930\times10^{-2}k+0.17945466\times10^{-2}k^{2}+0.13563368\times10^{-3}k^{3} \\
& +0.18084490\times10^{-5}k^{4})n^{6} \\
& +(-0.79560456\times10^{-3}-0.34308862\times10^{-3}k-0.41077629\times10^{-4}k^{2} \\
& -0.10333994\times10^{-5}k^{3})n^{7} \\
& +(0.27239764\times10^{-4}+0.66848028\times10^{-5}k+0.29064360\times10^{-6}k^{2})n^{8} \\
& +(-0.46018570\times10^{-6}-0.43058310\times10^{-7}k)n^{9}+0.26911444\times10^{-8}n^{10}
\end{aligned}
$$

$\Xi^{n,k}$[illegible]

$$
\begin{aligned}
= & -0.78404008\times10^{-1}k-0.12694784\times10k^{2}-0.20245809\times10k^{3}-0.11084864\times10k^{4}\\
& -0.16149724k^{5}+0.66966077\times10^{-1}k^{6}+0.31844075\times10^{-1}k^{7}+0.53865947\times10^{-2}k^{8}\\
& +0.41852678\times10^{-3}k^{9}+0.12400793\times10^{-4}k^{10}\\
& +(0.25468750-0.71828847k+0.69106213k^{2}+0.24526567\times10k^{3}+0.16668096\times10k^{4}\\
& +0.46308017k^{5}+0.47626410\times10^{-1}k^{6}-0.20290798\times10^{-2}k^{7}-0.72854662\times10^{-3}k^{8}\\
& -0.37202380\times10^{-4}k^{9})n\\
& +(0.10694353\times10+0.35087254\times10k+0.17139020\times10k^{2}-0.62379912k^{3}\\
& -0.64617284k^{4}-0.16860487k^{5}-0.16343858\times10^{-1}k^{6}-0.15811012\times10^{-3}k^{7}\\
& +0.40302579\times10^{-4}k^{8})n^{2}\\
& +(-0.22888366\times10-0.41517179\times10k-0.21096223\times10k^{2}-0.25393779k^{3}\\
& +0.72116426\times10^{-1}k^{4}+0.19986979\times10^{-1}k^{5}+0.12152778\times10^{-2}k^{6}\\
& -0.12400793\times10^{-4}k^{7})n^{3}\\
& +(0.14630790\times10+0.18402741\times10k+0.72115903k^{2}+0.89842903\times10^{-1}k^{3}\\
& -0.33264160\times10^{-2}k^{4}-0.10118272\times10^{-2}k^{5}-0.10850694\times10^{-4}k^{6})n^{4}\\
& +(-0.41443557-0.36505263k-0.97740512\times10^{-1}k^{2}-0.74428982\times10^{-2}k^{3}\\
& +0.20345052\times10^{-3}k^{4}+0.10850694\times10^{-4}k^{5})n^{5}\\
& +(0.58847301\times10^{-1}+0.34576246\times10^{-1}k+0.54675631\times10^{-2}k^{2}+0.15597873\times10^{-3}k^{3}\\
& -0.27126735\times10^{-5}k^{4})n^{6}\\
& +(-0.43146769\times10^{-2}-0.15183706\times10^{-2}k-0.10889447\times10^{-3}k^{2}\\
& -0.77504960\times10^{-6}k^{3})n^{7}\\
& +(0.15614827\times10^{-3}+0.26630220\times10^{-4}k+0.62972780\times10^{-6}k^{2})n^{8}\\
& +(-0.24099198\times10^{-5}-0.14532179\times10^{-6}k)n^{9}+0.12110150\times10^{-7}n^{10}
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{7,3} = & -0.32350054k-0.12196816\times10k^{2}-0.11188980\times10k^{3}+0.47317412\times10^{-1}k^{4}\\
& +0.39691297k^{5}+0.13732367k^{6}+0.46223958\times10^{-2}k^{7}-0.46502975\times10^{-2}k^{8}\\
& -0.74404761\times10^{-3}k^{9}-0.33068783\times10^{-4}k^{10}\\
& +(0.40491588+0.38817559k+0.25880466\times10k^{2}+0.32432107\times10k^{3}+0.91814125k^{4}\\
& -0.24745370k^{5}-0.14884982k^{6}-0.19051752\times10^{-1}k^{7}-0.12400793\times10^{-3}k^{8}\\
& +0.66137566\times10^{-4}k^{9})n\\
& +(-0.14538716+0.91073945k-0.17954750\times10k^{2}-0.28230622\times10k^{3}-0.96859199k^{4}\\
& -0.29492188\times10^{-1}k^{5}+0.27379919\times10^{-1}k^{6}+0.25545634\times10^{-2}k^{7}\\
& -0.24801587\times10^{-4}k^{8})n^{2}\\
& +(-0.14832795\times10-0.27600992\times10k+0.43922536\times10^{-2}k^{2}+0.10180582\times10k^{3}\\
& +0.31289785k^{4}+0.15248843\times10^{-1}k^{5}-0.20833333\times10^{-2}k^{6}-0.33068783\times10^{-4}k^{7})n^{3}\\
& +(0.18224281\times10+0.21452672\times10k+0.39862061k^{2}-0.13687021k^{3}\\
& -0.33320674\times10^{-1}k^{4}-0.34722222\times10^{-3}k^{5}+0.28935185\times10^{-4}k^{6})n^{4}\\
& +(-0.81098656-0.63701217k-0.92134603\times10^{-1}k^{2}+0.98289207\times10^{-2}k^{3}\\
& +0.97656250\times10^{-3}k^{4})n^{5}\\
& +(0.15778062+0.74965187\times10^{-1}k+0.50451207\times10^{-2}k^{2}-0.27126735\times10^{-3}k^{3}\\
& -0.72337963\times10^{-5}k^{4})n^{6}\\
& +(-0.13684986\times10^{-1}-0.32810432\times10^{-2}k-0.83705356\times10^{-4}k^{2}\\
& +0.20667989\times10^{-5}k^{3})n^{7}\\
& +(0.49583798\times10^{-3}+0.52509610\times10^{-4}k+0.38752480\times10^{-6}k^{2})n^{8}\\
& +(-0.70723276\times10^{-5}-0.25834987\times10^{-6}k)n^{9}+0.32293733\times10^{-7}n^{10}
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{6,4} = \; & -0.12924986k-0.62146538k^{2}-0.33739623k^{3}+0.61429059k^{4}+0.35770043k^{5} \\
& -0.35862222\times10^{-1}k^{6}-0.36192491\times10^{-1}k^{7}-0.20616319\times10^{-2}k^{8} \\
& +0.65104166\times10^{-3}k^{9}+0.57870370\times10^{-4}k^{10} \\
& +(0.23536965-0.10371433\times10^{-2}k+0.24034616\times10k^{2}+0.21481291\times10k^{3} \\
& -0.86575651k^{4}-0.70665804k^{5}-0.20437283\times10^{-1}k^{6}+0.28566261\times10^{-1}k^{7} \\
& +0.23871527\times10^{-2}k^{8}-0.57870370\times10^{-4}k^{9})n \\
& +(-0.44334491+0.58338647k-0.32966030\times10k^{2}-0.29531839\times10k^{3}+0.29861320k^{4} \\
& +0.37868110k^{5}+0.24553313\times10^{-1}k^{6}-0.34288194\times10^{-2}k^{7}-0.43402777\times10^{-4}k^{8})n^{2} \\
& +(-0.68031119-0.14565118\times10k+0.22746793\times10k^{2}+0.16168915\times10k^{3} \\
& -0.15959562\times10^{-1}k^{4}-0.61472800\times10^{-1}k^{5}-0.10416666\times10^{-2}k^{6} \\
& +0.57870370\times10^{-4}k^{7})n^{3} \\
& +(0.16702157\times10+0.14553675\times10k-0.76323796k^{2}-0.37312430k^{3} \\
& +0.13495550\times10^{-2}k^{4}+0.23708767\times10^{-2}k^{5}+0.72337963\times10^{-5}k^{6})n^{4} \\
& +(-0.10880642\times10-0.57201923k+0.10259433k^{2}+0.26450828\times10^{-1}k^{3} \\
& -0.13563368\times10^{-3}k^{4}-0.21701388\times10^{-4}k^{5})n^{5} \\
& +(0.28208466+0.75512131\times10^{-1}k-0.56822600\times10^{-2}k^{2}-0.58322482\times10^{-3}k^{3} \\
& +0.18084490\times10^{-5}k^{4})n^{6} \\
& +(-0.27000032\times10^{-1}-0.32197174\times10^{-2}k+0.11393229\times10^{-3}k^{2} \\
& +0.36168981\times10^{-5}k^{3})n^{7} \\
& +(0.96198188\times10^{-3}+0.48658583\times10^{-4}k-0.67816839\times10^{-6}k^{2})n^{8} \\
& +(-0.13054741\times10^{-4}-0.22605613\times10^{-6}k)n^{9}+0.56514033\times10^{-7}n^{10}
\end{aligned}
$$

$$
\begin{aligned}
\Xi^{n,k}_{5,5} = \; & -0.59377278k^{2}+0.77996013k^{4}-0.14597683k^{6}+0.65538194\times10^{-2}k^{8} \\
& -0.69444444\times10^{-4}k^{10} \\
& +(-0.31035156+0.33573174\times10k^{2}-0.20336137\times10k^{4}+0.19766710k^{6} \\
& -0.37326388\times10^{-2}k^{8})n \\
& +(0.85280599-0.56835340\times10k^{2}+0.16931440\times10k^{4}-0.69542100\times10^{-1}k^{6} \\
& +0.86805554\times10^{-4}k^{8})n^{2} \\
& +(-0.14760918\times10+0.42791076\times10k^{2}-0.53445096k^{4}+0.38194444\times10^{-2}k^{6})n^{3} \\
& +(0.18030287\times10-0.15599100\times10k^{2}+0.52585178\times10^{-1}k^{4}-0.43402777\times10^{-4}k^{6})n^{4} \\
& +(-0.11797414\times10+0.25571221k^{2}-0.14648438\times10^{-2}k^{4})n^{5} \\
& +(0.34297099-0.13679335\times10^{-1}k^{2}+0.10850694\times10^{-4}k^{4})n^{6} \\
& +(-0.33799235\times10^{-1}+0.24956597\times10^{-3}k^{2})n^{7} \\
& +(0.15401204\times10^{-2}-0.13563368\times10^{-5}k^{2})n^{8}+0.22718641\times10^{-4}n^{9} \\
& +0.67816840\times10^{-7}n^{10}
\end{aligned}
$$