Induction Equation - The Diffusion Limit

In this section, we take $R_m \ll 1$ and study pure magnetic diffusion by considering

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B}. \tag{2.33}$$

This is a diffusion equation and so indicates that any irregularities in an initial magnetic field will diffuse away and be smoothed out. The field will tend towards a simpler uniform field. This process of smoothing out variations will occur on a timescale given by the diffusion timescale, $\tau_{diffusion} \approx L^2/\eta$.

Example 2.4.1 If the radius of a sunspot is $l = 10^7 m$ and $\eta = 10^3 m^2 s^{-1}$, then the sunspot magnetic field will diffuse away in

$$\tau_{diffusion} \approx 10^{11} \text{seconds} \approx 3,000 \text{years}.$$

Consider the diffusion of a current sheet. Suppose the initial magnetic field is anti-parallel with

$$\mathbf{B} = B_0(x)\hat{\mathbf{i}}$$

where

$$B_{0(x)} = \begin{cases} B_0, & x > 0, \\ -B_0, & x < 0. \end{cases} \tag{2.34}$$

How does this intial discontinuity in the magnetic field at $x = 0$ diffuse away? Assuming that the magnetic field remains in the $\hat{\mathbf{i}}$ direction for all time so that $\mathbf{B} = B(x,t)\hat{\mathbf{i}}$, then

$$\frac{\partial B}{\partial t} = \eta \frac{\partial^2 B}{\partial x^2}. \tag{2.35}$$

This must be solved subject to the initial condition $B = B_0(x)$ at $t = 0$. Since the problem has no implicit lengthscale associated with it, that is there are no boundary conditions at finite values of x and $B_0(x)$ has no lengthscale in it, we look for a *similarity solution*. The similarity variable must be a dimensionless variable. Thus, we set $B = B(\xi)$ where

$$\xi = \frac{x}{(4\eta t)^{1/2}}.$$

ξ is dimensionless and the factor of 4 simplifies the algebra later on. Using this definition of ξ the derivatives with respect to x and t can be expressed as

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial \xi}{\partial x}\frac{d}{d\xi} = \frac{1}{(4\eta t)^{1/2}}\frac{d}{d\xi} \\ \frac{\partial^2}{\partial x^2} &= \frac{\partial \xi}{\partial x}\frac{d}{d\xi}\left(\frac{\partial}{\partial x}\right) = \frac{1}{4\eta t}\frac{d^2}{d\xi^2} \\ \frac{\partial}{\partial t} &= \frac{\partial \xi}{\partial t}\frac{d}{d\xi} = -\frac{1}{2}\frac{\xi}{t}\frac{d}{d\xi} \end{aligned}$$

Hence, the diffusion equation (2.35) becomes

$$\begin{aligned} \frac{\partial B}{\partial t} &= \eta \frac{\partial^2 B}{\partial x^2} \\ \Rightarrow -\frac{1}{2}\frac{\xi}{t}\frac{dB}{d\xi} &= \frac{\eta}{4\eta t}\frac{d^2 B}{d\xi^2} \\ \Rightarrow \frac{d^2 B}{d\xi^2} + 2\xi\frac{dB}{d\xi} &= 0. \end{aligned}$$

This is a first order, ordinary differential equation for $dB/d\xi$ that can be

solved either by separation or by the integrating factor method. Integrating once gives,

$$\frac{dB}{d\xi} = Ae^{-\xi^2},$$

where A is a constant. The final integration can be performed formally so that

$$B(x,t) = C + A\int_0^{\xi} e^{-u^2}du,$$

where C is another constant. Now using the conditions $B(x=0,t)=0$ and $B(x\to\pm\infty,t)=\pm B_0$, so that

$$C = 0 \text{ and } A = \frac{2B_0}{\sqrt{\pi}},$$

the final solution for the magnetic field is

$$B(x,t) = \frac{2B_0}{\sqrt{\pi}}\int_0^{x/(4\eta t)^{1/2}} e^{-u^2}du = B_0\text{erf}\left(\frac{x}{(4\eta t)^{1/2}}\right). \qquad (2.36)$$

In (2.36)

$$\text{erf}(\xi) = \frac{2}{\sqrt{\pi}}\int_0^{\xi} e^{-u^2}du,$$

is the *error function*.

Subsections

- Properties of the solution for B(x,t)

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Prof. Alan Hood