Electromagnetic Equations

The electromagnetic equations are given in MKS units but the magnetic field is frequently quoted in terms of Gauss, where 1 tesla = 10^4 Gauss. Maxwells's equations are

$$\nabla \times \mathbf{B} = \mu \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}, \tag{2.1}$$

where the last term on the right hand side is the displacement current.

$$\nabla \cdot \mathbf{B} = 0, \tag{2.2}$$

is the solenoidal condition for the magnetic induction, indicating that there are no magnetic monopoles. That is there are no sources and sinks for magnetic field lines. Next, we have Faraday's law of magnetic induction with

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.3}$$

showing that a spatially varying electric field can induce a magnetic field. Finally, charge conservation gives

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon}\rho^*. \tag{2.4}$$

In the above equations

- **B** is the magnetic induction, but is usually referred to as the magnetic field,
- **j** is the current density,
- **E** is the electric field,
- $\rho^* = e(n^+ - n^-)$ is the charge density, where n^+ is the number of ions and n^- is the number of electrons,
- ϵ is the permittivity of free space and

- μ is the magnetic permeability in a vacuum and is $4\pi \times 10^{-7}$ Hm^{-1}.

Auxiliary variables are the electric displacement $\mathbf{D} = \epsilon\mathbf{E}$ and the magnetic field $\mathbf{H} = \mathbf{B}/\mu$. Finally, the speed of light in a vacuum is

$c = (\mu\epsilon)^{-1/2} \approx 3 \times 10^8$ ms^{-1}.

Example 2.1.1 The displacement current in Ampere's law may be neglected if the typical plasma velocities are much less than the speed of light, c. To see this we assume that the typical lengthscale for plasma variations is L and that typical timescales are of order T. This simply means that L is the spatial distance over which quantities vary. Thus, if a quantity is given by $y = e^{-x/a}$, then y varies appreciably over a distance $x = a$. Thus, we would choose $L = a$ in this case. In a similar manner T is an estimate of the time needed for the plasma to change. These two quantities can be used to define a typical plasma velocity as $V = L/T$. This kind of approximation allows us to estimate the typical size of the terms in (2.3). Thus,

$$\nabla \times \mathbf{E} \approx \frac{E}{L} \text{ and } \frac{\partial \mathbf{B}}{\partial t} \approx \frac{B}{T}.$$

As the equation only has two terms they must be equal and so we obtain

$$E = \frac{L}{T}B = VB.$$

Now in (2.1) the left hand side is approximately B/L but the displacement current is

$$\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \approx \frac{1}{c^2}\frac{E}{T} = \frac{V}{c^2}\frac{B}{T} = \frac{B}{L}\frac{V}{c^2}\frac{L}{T} = \frac{B}{L}\frac{V^2}{c^2}.$$

Hence, if the typical plasma velocities satisfy $V^2 \ll c^2$ then the displacement current is much smaller than $\nabla \times \mathbf{B}$. This is the *MHD approximation*, so that Ampere's law simplifies to

$$\nabla \times \mathbf{B} = \mu \mathbf{j}. \tag{2.5}$$

Using similar ideas and by assuming the plasma obeys charge neutrality, $n^+ - n^- \ll n$, where n is the total number density, we can neglect the charge density in (2.4) provided

$$n \gg \frac{\epsilon BV}{eL}. \tag{2.6}$$

e is the charge on an electron.

The final electromagnetic equation is ohm's law,

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{2.7}$$

where σ is the electrical conductivity in mho m^{-1}. This is effectively a generalisation of the simple *voltage = current* times *resistance* to a moving conductor. $\mathbf{v}$ is the plasma velocity and it is Ohm's law that provides the link between the electromagnetic equations and the plasma fluid equations.

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