# Kinematic Concentration of the Field

Granulation provides direct evidence that convection is occurring beneath the photosphere. What is the effect of convection on the magnetic field that threads through the photospheric surface? Since the magnetic field is essentially frozen to the plasma, convective motions will tend to concentrate the magnetic field in the downdrafts and weaken it in the updrafts. This is shown schematically in Figure 2.7.

**Figure 2.7:** (a) Typical convection cells seem from the side. (b) A uniform magnetic field is carried to the downdrafts.

We can model granulation by an array of cells with a flow given by the velocity

$$\mathbf{v} = (v_x, 0 v_z),$$

where

$$v_x = v_0 \sin kx, \; v_z = -kz v_0 \cos kx. \tag{2.46}$$

To understand the nature of the flow, we analyse the velocity components at a few special points. For example, consider $z = 0$ at the photospheric surface. Here $v_z = 0$ and there is only a horizontal flow with $v_x = v_0 \sin kx$.



At $x = 0$ we have $v_x = 0$ and $v_z = -kv_0 z$ so that this is an updraft for $z < 0$.

At $x = \pi/k$ we have $v_x = 0$ again and $v_z = kv_0 z$. This is then the location of a downdraft if $z < 0$. Now we can ask what happens to an initially uniform, veritcal magnetic field in such a flow. To make analytical progress we neglect the back reaction of the field on the flow and only consider the induction equation. The initial configuration is shown in Figure 2.8.

**Figure 2.8:** A model of solar convection. The initial magnetic field is uniform and vertical.

Suppose that the magnetic field, $\mathbf{B}$, remains vertical for all $t$ so that,

$$\mathbf{B} = B(x, z, t)\mathbf{z}.$$

However, the magnetic field must satisfy the solenoidal condition, (2.2) so that

$$\nabla \cdot \mathbf{B} = \frac{\partial B}{\partial z} = 0.$$

This implies that $B$ must be independent of $z$. Thus,

$$\mathbf{B} = B(x, t)\mathbf{z}.$$

The induction equation is

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}).$$

and substituting in the expressions for $\mathbf{v}$ and $\mathbf{B}$ gives,

$$\mathbf{v} \times \mathbf{B} = (0, -v_x B, 0),$$

and

$$\nabla \times (\mathbf{v} \times \mathbf{B}) = \left[\frac{\partial}{\partial z}(v_x B), 0, -\frac{\partial}{\partial x}(v_x B)\right].$$

However, because of the form of $v_x$ and $B(x,t)$, we have

$$\frac{\partial}{\partial z}(v_x B) = 0.$$

This implies that there is only a $x$ component of the induction equation. Thus, (2.31) becomes

$$\frac{\partial B}{\partial t} + v_0 \sin kx \frac{\partial B}{\partial x} + v_0 k \cos kx B = 0. \tag{2.47}$$

This is a first order, linear, partial differential equation which we need to solve subject to the initial condition that the field is uniform, that is,

$$B(x, t = 0) = B_0. \tag{2.48}$$

We solve (2.47) using the *Method of Characteristics*. We build up to the solution of (2.47) through a few simple examples that will make the Method of Charactersitics more apparent.

**Example 2.6.1**

$$\mathbf{V} \cdot \nabla f = u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} + w \frac{\partial f}{\partial z},$$

gives the derivative of $f(x,y,z)$ in the direction of $\mathbf{V} = (u,v,w)$. Consider, for example, $\mathbf{V} = (1,-2,3)$ and $f(x,y,z) = x^2 + 2y + e^{-x}$. Then,

$$\mathbf{V} \cdot \nabla f = 2x - 4 - 3e^{-x}.$$

**Example 2.6.2** A curve in the $x-y$ plane can be expressed in parametric form as $x = x(s)$ and $y = y(s)$ in terms of the parameter $s$. This is shown in



Figure 2.9.

**Figure 2.9:** A parametric representation of a curve in the $x-y$ plane. The tangent to the curve is shown at the parameter value $s_1$.

The tangent to the curve defines a vector that gives the direction of the curve. The gradient of the tangent is

$$m = \frac{dy}{dx} = \frac{dy/ds}{dx/ds} = \frac{y\text{ - step}}{x\text{ - step}}.$$

Hence, a vector defining the direction of the curve is $\mathbf{V} = \left(\frac{dx}{ds}, \frac{dy}{ds}\right)$, since $m = \frac{dy/ds}{dx/ds}$.

**Example 2.6.3** If $u = u(x,y)$ and $x, y$ lie on the curve defined by $x = x(s)$, $y = y(s)$, then

$$u = u(x(s), y(s))$$

is just a function of $s$. By the chain rule,

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} = \mathbf{V} \cdot \nabla u.$$

This is the directional derivative of $u$ in the direction of $\mathbf{V}$ along the curve.

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**Subsections**


- Method of Characteristics

**Next:** Method of Characteristics **Up:** MHD Equations **Previous:** Induction Equation - Frozen-in

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