Induction Equation - Frozen-in Theorem

Now consider the usual solar limit with $R_m \gg 1$ so that (2.29) simplifies to (2.31), namely

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}).$$

The *frozen-flux theorem* of Alfven (1943) applies.

Alfvén's Theorem. `` In a perfectly conducting fluid ($R_m \to \infty$), magnetic field lines move with the fluid: the field lines are ` frozen' into the plasma. "

This theorem states that motions along the field lines do not change the field but motions transverse to the field carry the field with them.

Proof. The proof will be developed in several steps. Firstly, we need to make use of Gauss's divergence theorem, which is

$$\int_V \nabla \cdot \mathbf{A} dV = \int_S \mathbf{A} \cdot \mathbf{dS},$$

where S is a *closed* surface enclosing the volume V, and of Stokes' theorem

$$\int_S \nabla \times \mathbf{A} \cdot \mathbf{dS} = \int_C \mathbf{A} \cdot \mathbf{dl},$$

where C is a closed curve around the *open* surface S. In the above, $\mathbf{dS} = \hat{\mathbf{n}} dS$ where $\hat{\mathbf{n}}$ is the outward unit normal.

(i) From $\nabla \cdot \mathbf{B} = 0$, for all time, we may integrate over the volume of the plasma to deduce

$$0 = \int_V \nabla \cdot \mathbf{B} dV = \int_S \mathbf{B} \cdot \mathbf{dS}, \qquad \forall t, \tag{2.37}$$

for any *closed* surface S.

(ii) Next we consider the time behaviour of the *magnetic flux*, Φ, through a closed curve C, around an *open* surface S_1.

$$\Phi = \int_{S_1} \mathbf{B}(\mathbf{r}, t) \cdot \mathbf{dS}. \tag{2.38}$$

Now Φ changes in time because $\mathbf{B}$ changes in time and because the curve C changes in response to plasma motions.

(iii) Consider how the curve C moves with the fluid motion to the curve C' in the time interval δt.

Figure 2.3: Curve C at time t moves with the plasma motion to the curve C' at time $t + \delta t$.

The motion of the surface enclosed by curve C to the surface enclosed by curve C' generates a volume V enclosed by the surface S as depicted in Figure (2.3).

Figure 2.4: The volume V consists of the top surface, the bottom surface and the sides.

The volume V is enclosed by the closed surface S that is made up of the top surface, enclosed by C', the bottom surface enclosed by C and the sides.

(iv) Consider the total flux through the closed surface S in (iii). At time $t+\delta t$, when the magnetic field is $\mathbf{B}(\mathbf{r},t+\delta t)$, we have from (2.37)

$$\begin{aligned} 0 &= \int_{closed\,S} \mathbf{B}(\mathbf{r},t+\delta t)\cdot d\mathbf{S}, \\ &= \int_{top} \mathbf{B}(\mathbf{r},t+\delta t)\cdot d\mathbf{S} + \int_{bottom} \mathbf{B}(\mathbf{r},t+\delta t)\cdot d\mathbf{S} + \int_{side} \mathbf{B}(\mathbf{r},t+\delta t)\cdot d\mathbf{S}, \\ &= \int_{C'} \mathbf{B}(\mathbf{r},t+\delta t)\cdot d\mathbf{S} - \int_{C} \mathbf{B}(\mathbf{r},t+\delta t)\cdot d\mathbf{S} + \int_{side} \mathbf{B}(\mathbf{r},t+\delta t)\cdot d\mathbf{S}. \end{aligned} \quad (2.39)$$

(v) Consider the contribution to the total flux from the curved side.

Figure 2.5: The element of area on the curved side.

A small element of length on the curve C traces out the shaded region in Figure (2.5). Then $d\mathbf{S}$ is given by the outward normal, $\hat{\mathbf{n}}$, times the area of the shaded region. This area is approximately the area of the parallelogram with sides $d\mathbf{l}$ and $\mathbf{v}\delta t$. Hence, on the side

$$d\mathbf{S} = d\mathbf{l}\times\hat{\mathbf{n}}\delta t. \quad (2.40)$$

Thus, from (2.39) we have

$$0 = \int_{C'} \mathbf{B}(\mathbf{r},t+\delta t)\cdot d\mathbf{S} - \int_{C} \mathbf{B}(\mathbf{r},t+\delta t)\cdot d\mathbf{S} + \int_{C} \mathbf{B}(\mathbf{r},t+\delta t)\cdot d\mathbf{l}\times\mathbf{v}\delta t.$$

Hence,

$$\int_{C'} \mathbf{B}(\mathbf{r},t+\delta t)\cdot d\mathbf{S} = \int_{C} \mathbf{B}(\mathbf{r},t+\delta t)\cdot d\mathbf{S} - \delta t\int_{C} \mathbf{B}(\mathbf{r},t+\delta t)\cdot d\mathbf{l}\times\mathbf{v}, \quad (2.41)$$

so that the flux through the curve C', at time $t+\delta t$, is equal to the flux through the curve C minus the contribution from the sides.

(vi) How does Φ change in time? This is simply the difference between the value of Φ at time $t+\delta t$ and Φ at time t. Thus, the change in flux is

$$\begin{aligned} \delta\Phi &= \Phi(t+\delta t) \text{ through } C' - \Phi(t) \text{ through } C, \\ &= \int_{C'} \mathbf{B}(\mathbf{r},t+\delta t)\cdot d\mathbf{S} - \int_{C} \mathbf{B}(\mathbf{r},t)\cdot d\mathbf{S}. \end{aligned} \quad (2.42)$$

Now use (2.41) to give

$$\delta\Phi = \int_{C} \left[\mathbf{B}(\mathbf{r},t+\delta t) - \int_{C} \mathbf{B}(\mathbf{r},t)\right]\cdot d\mathbf{S} - \delta t\int_{C} \mathbf{B}(\mathbf{r},t+\delta t)\cdot d\mathbf{l}\times\mathbf{v}.$$

If δt is small then we can approximate the integrand in the surface integral, $\mathbf{B}(\mathbf{r},t+\delta t) - \int_C \mathbf{B}(\mathbf{r},t)$ by $\delta t\partial\mathbf{B}/\partial t$. Hence,

$$\delta\Phi = \delta t\int_{C} \frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{S} - \delta t\int_{C} \mathbf{v}\times\mathbf{B}\cdot d\mathbf{l}. \quad (2.43)$$

Here we have used the vector identity

$$\mathbf{B}\cdot(d\mathbf{l}\times\mathbf{v}) = \mathbf{v}\cdot(\mathbf{B}\times d\mathbf{l}) = (\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}.$$

(vii) The final step in the proof is to use the induction equation, for $R_m \gg 1$, i.e. (2.31). In this limit we have,

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}).$$

Thus, (2.43) becomes,

$$\begin{aligned}\frac{\delta\Phi}{\delta t} &= \int_C \nabla\times(\mathbf{v}\times\mathbf{B})\cdot d\mathbf{S} - \int_C \mathbf{v}\times\mathbf{B})\cdot d\mathbf{l},\\ &= \int_C \mathbf{v}\times\mathbf{B})\cdot d\mathbf{l} - \int_C(\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}, \text{ on using Stoke's theorem,}\\ &= 0.\end{aligned}$$

As $\delta t \to 0$ we have

$$\frac{\delta\Phi}{\delta t} \to \frac{d\Phi}{dt}.$$

Thus, we reach the conclusion that Φ does not change in time and so

$$\frac{d\Phi}{dt} = \frac{d}{dt}\left\{\int_C \mathbf{B}\cdot d\mathbf{S}\right\} = 0, \qquad (2.44)$$

where C is any closed contour moving with the fluid. The magnetic lines of force are *frozen into the fluid*.

Example 2.5.1 The identity of a *flux tube* is preserved by the motion.

Figure 2.6: A flux tube at time t_1 is deformed by the fluid motion at the later time t_2.

The same fluid occupies the interior of the flux tube at time t_2 as did at time the earlier time t_1. If the area of the flux tube is small, then the field strength will be approximately constant across the area of the tube and we obtain the important result that

$$B \times \text{ cross sectional area } = \Phi = \text{ a constant,}$$

so that

$$BA = \Phi. \qquad (2.45)$$

Therefore, if the area, A, is reduced by the fluid motion then the field strength, B, becomes stronger.

Next Up Previous

Next: Kinematic Concentration of the **Up:** MHD Equations **Previous:** Properties of the solution

Prof. Alan Hood

2000-01-11