Icarus **147**, 129–144 (2000)
doi:10.1006/icar.2000.6399, available online at http://www.idealibrary.com on IDEAL®

The Disturbing Function in Solar System Dynamics

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Received June 12, 1995; revised February 28, 2000

The planetary disturbing function is the basis of much analytical work in Solar System dynamics and series expansions of it that were derived in the last century are still in common use today. However, most previous expansions have the disadvantage of being in terms of the mutual inclination of the two masses. Also, several of the classical, high-order expansions contain a number of errors. A new algorithm for the derivation of the disturbing function in terms of the individual orbital elements of the two masses is presented. It allows the calculation, to any order, of the terms associated with any individual argument without the need for expanding the entire disturbing function. The algorithm is used to generate a new expansion which is complete to fourth-order in the eccentricities and inclinations, and incorporates a consistent numbering system for each argument. The properties of the expansion for a selected argument are discussed, and the use of the expansion is illustrated using the examples of the Titan–Hyperion 3:4 resonance and the possible Jupiter–Pallas 18:7 resonance.

Key Words: **celestial mechanics; orbits; resonances.**

1. INTRODUCTION

The Solar System is not just a collection of objects moving in arbitrary orbits. It has an intricate dynamical structure that has evolved over time. To understand the structure and evolution of the Solar System we must understand the consequences of Newton's universal law of gravitation. To do this we must understand the properties of the perturbing potential experienced by one object due to another; this is also known as the planetary disturbing function.

Despite its importance for planetary science, guidance on the properties and practical use of the disturbing function can only be found in the specialist literature on celestial mechanics. Our own experience is that when the disturbing function is encountered, many authors give resonant arguments and simply state that the terms in the semimajor axis are "of order unity." Typically a reference is given to a textbook on celestial mechanics without any way of knowing which particular arguments are being considered. Invariably these texts fail to provide a clear presentation, even allowing for inconsistencies in their approach and their use of a mutual inclination. In an era when, for example, the locations of resonant structures in planetary rings can be determined to within $\ll$1 km, it seems unusual that there is no consistent approach to the use of the disturbing function given that, effectively, it determines the location of the resonant feature. Our discovery of a new algorithm to facilitate the derivation of the terms associated with an explicit argument and its subsequent application to the generation of a fourth-order expansion makes it appropriate to combine the two by writing a paper incorporating these results in a form suitable for a nonspecialist readership. Therefore the aims of this paper are (i) to present an efficient algorithm to calculate the terms associated with a given argument and (ii) to present a completely new expansion of the disturbing function in a systematic form together with examples of its use. In this section we provide some basic definitions and an outline of previous work on the disturbing function.

Consider a mass m orbiting a primary of mass M_c in an elliptical path. Let the orbiting mass have position vector $\mathbf{r}$ relative to M_c and assume that the gravitational effect of the primary arises from a point mass (see Fig. 1). This problem is integrable and the orbital elements a, e, I, ϖ, and Ω, which denote semimajor axis, eccentricity, inclination, longitude of pericenter, and longitude of ascending node, respectively, of the mass m are constant. Consider now a third mass, m', with position vector $\mathbf{r}'$ relative to M_c and orbital elements a', e', I', ϖ', and Ω'. Assume $r < r'$ always. The mutual gravitational force between the two orbiting masses m and m' results in accelerations in addition to the standard two-body accelerations due to M_c (see Fig. 1). These additional accelerations of the secondary masses with respect to the primary can be obtained from the gradient of the disturbing function.

The equations of motion of the two orbiting masses can be written as

$$\ddot{\mathbf{r}} = \nabla(U + \mathcal{R}) \qquad \text{and} \qquad \ddot{\mathbf{r}}' = \nabla'(U' + \mathcal{R}'), \tag{1}$$

where

$$U = \mathcal{G}\frac{(M_c + m)}{r} \qquad \text{and} \qquad U' = \mathcal{G}\frac{(M_c + m')}{r'} \tag{2}$$

are the central, or two-body parts of the total potential and it is understood that the ∇ and ∇' operators denote that the gradient is with respect to the coordinates of the mass m and m',

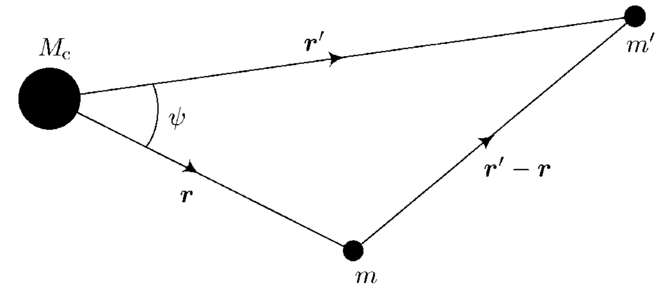

FIG. 1. The position vectors, $\mathbf{r}$ and $\mathbf{r}'$ of the two masses m and m' with respect to the central mass M_c. The angle between the position vectors is ψ.

respectively. The $\mathcal{R}$ and $\mathcal{R}'$ terms in Eq. (1) are the disturbing functions which represent the potentials arising from the gravitational effects of the external and internal masses respectively. These are given by

$$\mathcal{R} = \frac{\mathcal{G}m'}{|\mathbf{r}' - \mathbf{r}|} - \mathcal{G}m' \frac{\mathbf{r} \cdot \mathbf{r}'}{r'^3} \tag{3}$$

$$\mathcal{R}' = \frac{\mathcal{G}m}{|\mathbf{r} - \mathbf{r}'|} - \mathcal{G}m \frac{\mathbf{r} \cdot \mathbf{r}'}{r^3}. \tag{4}$$

The first term in each of these expressions is called the direct term while the second term, which arises from the choice of the origin of the coordinate system, is called the indirect term. If the origin of the coordinate system is at the center of mass, then there are no indirect terms.

At this stage we should point out one peculiarity of the disturbing function as used throughout this paper. The equation for the perturbing force is written as $\mathbf{F} = \nabla\mathcal{R}$ rather than the more usual $\mathbf{F} = -\nabla\mathcal{R}$, because $\mathcal{R}$ is actually the negative of the true potential. This is an historical quirk and has no effect on the inherent dynamics of the system.

It is clear that it is a trivial task to write $\mathcal{R}$ and $\mathcal{R}'$ in terms of the standard cartesian coordinates of each orbiting mass. The cartesian equations of motion of the inner perturbed body can be written in terms of the gradient of U and $\mathcal{R}$ as in Eq. (1) and numerical techniques can be used to study the orbit. However, in many problems it is often more useful to consider the variation of the orbital elements of the perturbed body with time. Hence we must express the relevant disturbing function in terms of the orbital elements of each body. Such an expansion is achieved by expanding $\mathcal{R}$ or $\mathcal{R}'$ as infinite series of the form

$$\mathcal{R} = \mu' \sum S(a, a', e, e', I, I') \cos\phi \tag{5}$$

and

$$\mathcal{R}' = \mu \sum S'(a, a', e, e', I, I') \cos\phi, \tag{6}$$

where $\mu = \mathcal{G}m$, $\mu' = \mathcal{G}m'$, and

$$\phi = j_1\lambda' + j_2\lambda + j_3\varpi' + j_4\varpi + j_5\Omega' + j_6\Omega. \tag{7}$$

The j_i are all integers, and λ and λ' denote the mean longitudes of each mass. We define the order of the argument as

$$N = |j_1 + j_2|. \tag{8}$$

As we shall see in Sect. 3 below, the value of N is the same as the lowest power of e, e', I, and I' that occurs in S or S'. Thus, when we refer to an Nth-order expansion we mean an expansion that includes all possible arguments up to and including order N, together with the corresponding terms in powers of the eccentricities or inclinations up to and including the Nth power.

The literature on the disturbing function has a distinguished history. The first paper to be published in the *Astronomical Journal* contained the start of a sixth-order expansion by Peirce (1849). Le Verrier (1855) produced a seventh-order expansion which was subsequently extended to eighth-order by Boquet (1889). Newcomb (1895) carried out work on a seventh-order expansion, while Norén and Wallberg (1899) produced a second-order expansion in canonical elements. Brown and Shook (1933) gave a clear derivation of the expansion to second-order in the keplerian orbital elements. In more recent times the expansion contained in Brouwer and Clemence (1961) has become widely adopted as a standard, low-order expansion; it is complete to third-order in all terms but contains some fourth-order terms associated with particular arguments.

In the expressions for S and S' in Eqs. (5) and (6) in each of these expansions the dependence on a and a' is given by means of Laplace coefficients (functions of $\alpha = a/a'$) and their derivatives (see, for example, Brouwer and Clemence 1961 and Sect. 2 below). The compact notation afforded by the use of Laplace coefficients means that they can be handled easily in calculations involving the disturbing function.

Another common but undesirable characteristic of all of these expansions is the use of angles referred to the mutual node of the orbiting masses, and the mutual inclination, $\mathcal{J}$, of the orbits given by

$$\cos\mathcal{J} = \cos I \cos I' + \sin I \sin I' \cos(\Omega' - \Omega). \tag{9}$$

This has the advantage of making the resulting size of the expansions more manageable, but it has the disadvantage of making studies of the effect of perturbations on individual inclinations or nodes more difficult.

As well as their consistent use of a mutual inclination there is a more serious drawback to some of these expansions—they contain errors. Most are typographical mistakes that are easy to identify and correct. For example, following the publication of the expansion by Le Verrier (1855), several editions of the *Annales de l'Observatoire de Paris* contained corrections to Le Verrier's original text. Murray (1985) pointed out a single sign error in Le Verrier's work that had not been corrected. The early printings of Brouwer and Clemence (1961) contained errors in the expansion that were corrected only with the third printing in 1971.

Using mathematical techniques from geophysics and artificial satellite theory, Kaula (1962) developed a form of the expansions using Legendre polynomials. This form of the expansion had the advantage that one could specify an argument of interest, ϕ say, and then isolate only those terms associated with that particular argument, avoiding the need to carry out a complete expansion. Another advantage was that because the dependence on e, e', I, and I' was given in the form of explicit functions, expansions to any order could be generated for the terms associated with any particular argument. This made Kaula's expansion ideal for use in a number of fields (see, for example, Allan 1969, 1970, Murray 1982, Dermott and Murray 1983). However, one disadvantage of Kaula's work was the absence of Laplace coefficients—the dependence on a and a' appeared as a summation in powers of α and reduced the compactness and ease of use of the expansion.

It is important to recognize that the "classical" expansions of the last century are still in use today. For example, Goldreich and Nicholson (1977) used the expansion by Peirce (1849), while Wisdom (1982), Murray (1986), and Šidlichovský and Melendo (1986) all used the expansion by Le Verrier (1855) in their work on algebraic mappings for asteroid motion at resonance.

In one of the first applications of computer algebra to celestial mechanics Deprit *et al.* (1971) discovered mistakes in Delaunay's lunar theory. Since then a number of authors have developed their own software packages for the development of series (see the review by Henrard 1989). For example, Broucke and Smith (1971) (and the references therein) describe a technique for computing the expansion of the disturbing function. Brumberg (1995) made use of a generic Poisson series processor to show how the disturbing function can be expanded; similar methods were employed by Laskar (1991) using Poincaré variables.

The need for correct, higher-order expansions and the ability to perform complicated, error-free series manipulation by computer, led Murray and Harper (1993) to generate an eighth-order expansion of the disturbing function in the individual orbital elements. The expansion was first produced using *Mathematica* (Wolfram 1991) and written as a file of integers that was then compared with a similar file produced independently using *Maple*. As a final check the expansion was compared with Kaula's form of the expansion for a large number of explicit cases. Details of the derivation are given in Harper and Murray (1994).

Our own experience suggests that many members of the Solar System dynamics community find the existing forms of the expansion difficult to derive and use. Therefore one of the goals of this paper is to present a new algorithm for the determination of the terms associated with a given argument in the expansion of the planetary disturbing function. This obviates the need for a complete series expansion involving all arguments up to and including a specified order. The expressions for each term involve e, e', I, and I' (instead of the mutual inclination, $\mathcal{J}$) as well as explicit Laplace coefficients (instead of infinite series in the ratio of the semimajor axes) and can be calculated up to any desired power of the eccentricities and inclinations. We present the algorithm in the next section and in Section 3 we provide detailed information on the properties of the expansion. In Section 4 we illustrate the use of the expansion by giving examples of how the appropriate terms are calculated for two specific cases in Solar System dynamics. A complete fourth-order expansion of the direct and indirect parts of the disturbing function is given in the Appendix; each argument in this expansion has been given a unique identification number for ease of reference. Some of the concepts and results developed in this paper, including the fourth-order expansion are made use of by Murray and Dermott (1999) where the disturbing function and its properties are examined in some detail. Murray and Dermott also derive an expansion (from first principles) of the disturbing function complete to second-order in the individual elements, and provide several analytical and numerical comparisons.

2. EXPANSION OF THE DISTURBING FUNCTION

It is unlikely that anyone making use of the planetary disturbing function requires all the arguments in the expansion; in practice the terms associated with only a few arguments are required. Therefore, although we give a complete fourth-order expansion in the Appendix for reference purposes, our primary objective is to produce a new form of the expansion which retains the advantages of Kaula's (1962) approach but expressed in terms of Laplace coefficients and their derivatives. Full details of the procedure for accomplishing this new derivation will be published elsewhere (K. Ellis, in preparation). Here we confine ourselves to giving the explicit form of the series.

From Eqs. (3) and (4) it is clear that we can write the disturbing function due to an external perturber as

$$\mathcal{R} = \frac{\mu'}{a'}(\mathcal{R}_{\mathrm{D}} + \alpha\mathcal{R}_{\mathrm{E}}) \tag{10}$$

and that due to an internal perturber as

$$\mathcal{R}' = \frac{\mu}{a}\left(\alpha\mathcal{R}_{\mathrm{D}} + \frac{1}{\alpha}\mathcal{R}_{\mathrm{I}}\right), \tag{11}$$

where $\alpha = a/a'$ is the ratio of the semimajor axes of the two orbits, and $\mu = \mathcal{G}m$, $\mu' = \mathcal{G}m'$. The direct part in both cases is given by

$$\mathcal{R}_{\mathrm{D}} = \frac{a'}{\Delta}, \tag{12}$$

with $\Delta^2 = r^2 + r'^2 - 2rr'\cos\psi$. Since

$$\mathbf{r}\cdot\mathbf{r}' = rr'\cos\psi, \tag{13}$$

where ψ is the angle between the two position vectors (see Fig. 1), we have

$$\mathcal{R}_{\mathrm{E}} = -\frac{r}{a}\frac{a'^2}{r'^2}\cos\psi \tag{14}$$

for the indirect part in the case of an external perturber and

$$\mathcal{R}_{\mathrm{I}} = -\frac{r'}{a'}\frac{a^2}{r^2}\cos\psi \tag{15}$$

for the indirect part in the case of an internal perturber.

We find that the series for the direct part, $\mathcal{R}_{\mathrm{D}}$ is given by

$$\begin{aligned}
\mathcal{R}_{\mathrm{D}} = {} & \sum_{i=0}^{\infty}\frac{(2i)!}{i!}\frac{(-1)^i}{2^{2i+1}}\alpha^i \\
& \times \sum_{j=-\infty}^{+\infty}\left\{\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\sum_{k=0}^{\ell}\binom{\ell}{k}(-1)^k\alpha^\ell\frac{\mathrm{d}^\ell}{\mathrm{d}\alpha^\ell}b_{i+\frac{1}{2}}^{(j)}(\alpha)\right\} \\
& \times \sum_{s=0}^{i}\sum_{l=0}^{i-s}\frac{(-1)^s 2^{2s}}{(i-s-l)!\,l!}\sum_{n=0}^{[s/2]}\frac{(2s-4n+1)(s-n)!}{2^{2n}n!(2s-2n+1)!} \\
& \times \sum_{m=0}^{s-2n}\kappa_m\frac{(s-2n-m)!}{(s-2n+m)!}\sum_{p,p'=0}^{s-2n}F_{s-2n,m,p}(I)\,F_{s-2n,m,p'}(I') \\
& \times \sum_{q,q'=-\infty}^{+\infty}X_{i+j-2l-2n-2p+q}^{i+k,\,i+j-2l-2n-2p}(e)\,X_{i+j-2l-2n-2p'+q'}^{-(i+k+1),\,i+j-2l-2n-2p'}(e') \\
& \times \cos[(i+j-2l-2n-2p'+q')\lambda' - (i+j-2l-2n \\
& - 2p+q)\lambda - q'\varpi' + q\varpi + (m-s+2n+2p')\Omega' \\
& - (m-s+2n+2p)\Omega],
\end{aligned} \tag{16}$$

where $\kappa_m = 1$ if $m = 0$ and $\kappa_m = 2$ if $m > 0$. The series for the indirect parts, $\mathcal{R}_{\mathrm{E}}$ and $\mathcal{R}_{\mathrm{I}}$ are given by

$$\begin{aligned}
\mathcal{R}_{\mathrm{E}} = {} & -\sum_{m=0}^{1}\kappa_m\frac{(1-m)!}{(1+m)!}\sum_{p,p'=0}^{1}F_{1,m,p}(I)\,F_{1,m,p'}(I') \\
& \times \sum_{q,q'=-\infty}^{+\infty}X_{1-2p+q}^{1,1-2p}(e)\,X_{1-2p'+q'}^{-2,1-2p'}(e') \\
& \times \cos[(1-2p'+q')\lambda' - (1-2p+q)\lambda - q'\varpi' + q\varpi \\
& -(1-2p'-m)\Omega' + (1-2p-m)\Omega]
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
\mathcal{R}_{\mathrm{I}} = {} & -\sum_{m=0}^{1}\kappa_m\frac{(1-m)!}{(1+m)!}\sum_{p,p'=0}^{1}F_{1,m,p}(I)\,F_{1,m,p'}(I') \\
& \times \sum_{q,q'=-\infty}^{+\infty}X_{1-2p+q}^{-2,1-2p}(e)\,X_{1-2p'+q'}^{1,1-2p'}(e') \\
& \times \cos[(1-2p'+q')\lambda' - (1-2p+q)\lambda - q'\varpi' + q\varpi \\
& -(1-2p'-m)\Omega' + (1-2p-m)\Omega].
\end{aligned} \tag{18}$$

In the expression for $\mathcal{R}_{\mathrm{D}}$ the coefficients $b_{i+\frac{1}{2}}^{(j)}(\alpha)$ are Laplace coefficients (see, for example, Brouwer and Clemence 1961) given by

$$\frac{1}{2}b_s^{(j)}(\alpha) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\cos j\varphi\,\mathrm{d}\varphi}{(1-2\alpha\cos\varphi+\alpha^2)^s}, \tag{19}$$

where s is a half integer. They can also be written in series form as

$$\begin{aligned}
\frac{1}{2}b_s^{(j)}(\alpha) = {} & \frac{s(s+1)\cdots(s+j-1)}{1\cdot 2\cdot 3\cdots j}\alpha^j\left[1+\frac{s(s+j)}{1(j+1)}\alpha^2\right. \\
& \left. + \frac{s(s+1)(s+j)(s+j+1)}{1\cdot 2(j+1)(j+2)}\alpha^4+\cdots\right]
\end{aligned} \tag{20}$$

or, equivalently in terms of elliptical integrals or hypergeometric functions. In the special case where $j = 0$ the factor outside the brackets in Eq. (20) is equal to unity. It can be shown that the series forms of the Laplace coefficient and its derivatives are always convergent for $\alpha < 1$. Note that there are no Laplace coefficients in $\mathcal{R}_{\mathrm{E}}$ and $\mathcal{R}_{\mathrm{I}}$.

The functions of inclination in all three expressions are defined by

$$\begin{aligned}
F_{s-2n,m,p}(I) = {} & \frac{1}{2^{s-2n}(s-2n)!} \\
& \times \sum_{t=0}^{\min(p,[(s-2n-m)/2])}\frac{(2s-4n-2t)!}{(s-2n-m-2t)!}\binom{s-2n}{t} \\
& \times \sin^{s-2n-m-2t} I\sum_{g=0}^{m}\binom{m}{g}\frac{\cos^g I}{2^{s-2n-2t}} \\
& \times \sum_{c=\max(0,p-t-m+g)}^{\min(p-t,s-2n-m-2t+g)}\binom{s-2n-m-2t+g}{c} \\
& \times \binom{m-g}{p-t-c}(-1)^{c-[(s-2n-m)/2]},
\end{aligned} \tag{21}$$

where $s - 2n = 1$ in $\mathcal{R}_{\mathrm{E}}$ and $\mathcal{R}_{\mathrm{I}}$ and the square brackets denote the integer part of the enclosed expression.

The functions of eccentricity, $X_c^{a,b}(e)$, in all three expressions are Hansen coefficients (see, for example, Plummer 1918, Jarnigan 1965, and Hughes 1981) which are defined by

$$X_c^{a,b}(e) = e^{|c-b|}\sum_{\sigma=0}^{\infty}X_{\sigma+\alpha,\sigma+\beta}^{a,b}\,e^{2\sigma}, \tag{22}$$

where, in this context, $\alpha = \max(0, c-b)$, $\beta = \max(0, b-c)$, and the $X_{c,d}^{a,b}$ are Newcomb operators which can be defined recursively by

$$X_{0,0}^{a,b} = 1 \tag{23}$$

$$X_{1,0}^{a,b} = b - a/2. \tag{24}$$

If $d = 0$, then

$$4cX^{a,b}_{c,0} = 2(2b - a)X^{a,b+1}_{c-1,0} + (b - a)X^{a,b+2}_{c-2,0}. \tag{25}$$

If $d \neq 0$, then

$$\begin{aligned} 4dX^{a,b}_{c,d} &= -2(2b + a)X^{a,b-1}_{c,d-1} - (b + a)X^{a,b-2}_{c,d-2} \\ &\quad - (c - 5d + 4 + 4b + a)X^{a,b}_{c-1,d-1} \\ &\quad + 2(c - d + b)\sum_{j\geq 2}(-1)^j \binom{3/2}{j} X^{a,b}_{c-j,d-j}. \end{aligned} \tag{26}$$

If $c < 0$ or $d < 0$, then $X^{a,b}_{c,d} = 0$. If $d > c$ then $X^{a,b}_{c,d} = X^{a,-b}_{d,c}$.

A fourth-order expansion of the direct and indirect parts of the disturbing function was generated by implementing the above algorithm in *Mathematica* (Wolfram 1991). A list of possible arguments was generated for each order (0, 1, 2, 3, 4) and then the terms associated with each argument were calculated. The resulting file of integers was compared with a similar expansion obtained by carrying out a full expansion using the technique employed by Murray and Harper (1993); the expansions agreed in all respects. Generating a full expansion using the new algorithm was found to be an order of magnitude faster than using the older method. The full expansions of $\mathcal{R}_{\mathrm{D}}$, $\mathcal{R}_{\mathrm{E}}$, and $\mathcal{R}_{\mathrm{I}}$ are given in the Appendix.

3. USE OF THE DISTURBING FUNCTION

The expansion of the direct and indirect parts of the disturbing function given in Eqs. (16)–(18) appears to be compact but at first glance its structure and the number of summations is intimidating. However, it is important to bear in mind that in a given application we are only interested in generating the terms associated with particular arguments.

Before showing how the number of summations can be reduced, we demonstrate the relationship between the integer coefficients in the cosine argument and the powers of the eccentricity and inclination in the corresponding term. Comparing Eq. (7) with the cosine argument in Eq. (16) gives the following relationships between the integers:

$$j_1 = (s - 2n - 2p' + q') + (i + j - 2l - s) \tag{27}$$

$$j_2 = -(s - 2n - 2p + q) - (i + j - 2l - s) \tag{28}$$

$$j_3 = -q' \tag{29}$$

$$j_4 = q \tag{30}$$

$$j_5 = m - (s - 2n - 2p') \tag{31}$$

$$j_6 = -m + (s - 2n - 2p), \tag{32}$$

where we have added and subtracted a term in s in the expressions for j_1 and j_2. Because all the angles in the cosine argument are longitudes, the argument satisfies the d'Alembert rule, whereby

$$\sum_{i=1}^{6} j_i = 0. \tag{33}$$

Hamilton (1994) has investigated the d'Alembert rule and similar properties in his comparison of Lorentz resonances with planetary gravitational resonances and satellite gravitational resonances.

From Eq. (22) we see that the lowest power of e arising from the Hansen coefficient $X^{a,b}_c(e)$ is $|c-b|$. Therefore, by inspection of the Hansen coefficients in e and e' in Eq. (16) we see that

$$X^{i+k,i+j-2l-2n-2p}_{i+j-2l-2n-2p+q}(e) = \mathcal{O}\big(e^{|q|}\big) \tag{34}$$

and

$$X^{-(i+k+1),i+j-2l-2n-2p'}_{i+j-2l-2n-2p'+q'}(e') = \mathcal{O}\big(e'^{|q'|}\big). \tag{35}$$

Therefore one property of the disturbing function is that the lowest powers of e and e' are the absolute values of the coefficients of ϖ and ϖ', respectively, in the cosine argument.

Similarly, an inspection of the inclination function defined in Eq. (21) shows that

$$F_{s-2n,m,p}(I) = \mathcal{O}\big(\sin^{|m-s+2n+2p|} I\big) \tag{36}$$

and

$$F_{s-2n,m,p'}(I') = \mathcal{O}\big(\sin^{|m-s+2n+2p'|} I'\big). \tag{37}$$

Therefore the lowest powers of $\sin I$ and $\sin I'$ that occur in Eq. (16) are the absolute values of the coefficients of Ω and Ω', respectively, in the cosine argument.

Now consider the various summations inherent in Eq. (16). By adding Eq. (31) and Eq. (32) we have

$$j_5 + j_6 = 2p' - 2p \tag{38}$$

and so $j_5 + j_6$ is always even. This places another constraint on the permissible arguments in the expansion, in addition to the constraint imposed by the d'Alembert relation. It also implies that the sum of the powers of the inclinations is always an even number and hence that the expansion can never contain an isolated, single power of an inclination. This property has also been investigated by Hamilton (1994).

An inspection of the various integers involved in the summations in Eq. (16) allows us to reduce the range of several of the summations. Let $N_{\max}$ be the maximum order of the expansion; this is the maximum sum of the powers of e, e', I, and I' that is desired in any one term associated with a given argument. Note that we must have $N_{\max} \geq N$, where N is the order of the argument as defined in Eq. (8). The following relationships hold throughout the calculation,

$$q = j_4 \tag{39}$$

$$q' = -j_3 \tag{40}$$

$$\ell_{\max} = N_{\max} - |j_5| - |j_6| \tag{41}$$

$$p_{\min} = -(j_5 + j_6)/2, \quad p'_{\min} = 0 \quad \text{if} \quad j_5 + j_6 < 0 \tag{42}$$

$$p_{\min} = 0, \qquad p'_{\min} = (j_5 + j_6)/2 \quad \text{if} \quad j_5 + j_6 \geq 0 \tag{43}$$

$$s_{\min} = \max(p_{\min},\ p'_{\min},\ j_6 + 2p_{\min},\ -j_5 + 2p'_{\min}) \tag{44}$$

$$i_{\max} = [(N_{\max} - |j_3| - |j_4|)/2], \tag{45}$$

where, as before, the square brackets in Eq. (45) denote the integer part of the expression.

As well as these global relationships, there are a number of intermediate definitions required for the summation. These are

$$n_{\max} = [(s - s_{\min})/2] \tag{46}$$

$$m_{\min} = 0 \quad \text{if} \quad s,\ j_5 \text{ are both even or both odd} \tag{47}$$

$$m_{\min} = 1 \quad \text{if} \quad s,\ j_5 \text{ are neither both even nor both odd} \tag{48}$$

$$p = (-j_6 - m + s - 2n)/2 \quad \text{with} \quad p \leq s - 2n \quad \text{and} \quad p \geq p_{\min} \tag{49}$$

$$p' = (j_5 - m + s - 2n)/2 \quad \text{with} \quad p' \leq s - 2n \quad \text{and} \quad p' \geq p'_{\min} \tag{50}$$

$$j = |j_2 + i - 2l - 2n - 2p + q|. \tag{51}$$

Again the square brackets denote the integer part of the expression.

Note that q and q' are determined directly from the coefficients of ϖ and ϖ' in the argument of interest, and remain fixed over all the summations. This removes the infinite summation over all values of q and q' in Eq. (16). On the other hand, p and p' change with s, n, and m but the relationships given in Eqs. (49) and (50) always hold.

We can now rewrite Eq. (16) as

$$\begin{aligned}\mathcal{R}_{\mathrm{D}} = &\sum_{i=0}^{i_{\max}} \frac{(2i)!}{i!} \frac{(-1)^i}{2^{2i+1}} \alpha^i \\ &\times \sum_{s=s_{\min}}^{i} \sum_{n=0}^{n_{\max}} \frac{(2s-4n+1)(s-n)!}{2^{2n} n!(2s-2n+1)!} \sum_{m=0}^{s-2n} \kappa_m \frac{(s-2n-m)!}{(s-2n+m)!} \\ &\times F_{s-2n,m,p}(I)\, F_{s-2n,m,p'}(I') \sum_{l=0}^{i-s} \frac{(-1)^s 2^{2s}}{(i-s-l)!l!} \\ &\times \sum_{\ell=0}^{\ell_{\max}} \frac{(-1)^\ell}{\ell!} \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k \alpha^\ell \frac{\mathrm{d}^\ell}{\mathrm{d}\alpha^\ell} b_{i+\frac{1}{2}}^{(j)}(\alpha) \\ &\times X_{-j_2}^{i+k,-j_2-j_4}(e)\, X_{j_1}^{-(i+k+1),\,j_1+j_3}(e') \\ &\times \cos[j_1\lambda' + j_2\lambda + j_3\varpi' + j_4\varpi + j_5\Omega' + j_6\Omega],\end{aligned} \tag{52}$$

where, as before, $\kappa_m = 1$ if $m = 0$ and $\kappa_m = 2$ if $m > 0$.

In the form of the expansion for $\mathcal{R}_{\mathrm{D}}$ given in Eq. (52) above, it is understood that the summations involved in the definitions of the functions of eccentricity and inclination need only be evaluated to a finite order which is at most equal to $N_{\max}$. The Hansen coefficient in e need only include terms up to order $N_{\max} - |j_3| - |j_5| - |j_6|$ in e; similarly the Hansen coefficient in e' need only include terms up to order $N_{\max} - |j_4| - |j_5| - |j_6|$ in e'. The F inclination function in I need only include terms up to order $N_{\max} - |j_3| - |j_4| - |j_5|$ in I; similarly the F function in I' need only include terms up to order $N_{\max} - |j_3| - |j_4| - |j_6|$ in I'.

In producing the truncated form of the full expression for $\mathcal{R}_{\mathrm{D}}$ we have also replaced the upper limit on the summation in ℓ from $\ell = \infty$ to $\ell = \ell_{\max} \neq \infty$, where $\ell_{\max}$ is defined in Eq. (41). This is not part of the general truncation process; it arises from the fact that for values of $\ell > \ell_{\max}$ all additional contributions to the summation are zero. This is because

$$\sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k f(k) = 0, \tag{53}$$

where $f(x)$ is any polynomial of degree less than ℓ.

Most previous expansions did not involve any $\sin I$ or $\cos I$ terms in the inclination parts. Instead the expansions were carried out in terms of $\sin \frac{1}{2} I$, or, to be more precise, $\sin \frac{1}{2} \mathcal{J}$. We can follow this example by noting that $\sin\, I = 2s\sqrt{1-s^2}$ and $\cos\, I = 1 - 2s^2$ where $s = \sin \frac{1}{2} I$ and make the appropriate substitutions in the definition of the F inclination function given in Eq. (21). This requires an additional series expansion in s (and s') and permits a direct comparison with the expansion of Murray and Harper (1993).

Now consider the indirect part for an external perturber (see Eq. (17)). A comparison of Eqs. (7) and (17) gives the following relationships between the integers:

$$j_1 = 1 - 2p' + q' \tag{54}$$

$$j_2 = -(1 - 2p + q) \tag{55}$$

$$j_3 = -q' \tag{56}$$

$$j_4 = q \tag{57}$$

$$j_5 = -(1 - 2p' - m) \tag{58}$$

$$j_6 = 1 - 2p - m. \tag{59}$$

An analysis of the integers involved in the expansion of this indirect part gives the following relationships:

$$q = j_4 \tag{60}$$

$$q' = -j_3 \tag{61}$$

$$p = (j_2 + j_4 + 1)/2 \tag{62}$$

$$p' = -(j_1 + j_3 - 1)/2 \tag{63}$$

$$m = j_5 - 2p' + 1 \tag{64}$$

and the expansion itself can now be written

$$\mathcal{R}_{\mathrm{E}} = -\kappa_m \frac{(1-m)!}{(1+m)!} \times F_{1,m,p}(I) F_{1,m,p'}(I') X_{-j_2}^{1,-j_2-j_4}(e) X_{j_1}^{-2,j_1+j_3}(e') \times \cos[j_1\lambda' + j_2\lambda + j_3\varpi' + j_4\varpi + j_5\Omega' + j_6\Omega], \quad (65)$$

where each of the quantities p, p', and m must be integers and equal to 0 or 1. If these conditions are not satisfied then the given argument does not appear in the expansion of the indirect part. As with the direct part we can reduce the extent of the series expansions in powers of the eccentricity and inclination, and the same modifications apply.

The equivalent expression for the indirect part due to an internal perturber is

$$\mathcal{R}_{\mathrm{I}} = -\kappa_m \frac{(1-m)!}{(1+m)!} \times F_{1,m,p}(I) F_{1,m,p'}(I') X_{-j_2}^{-2,-j_2-j_4}(e)\, X_{j_1}^{1,j_1+j_3}(e') \times \cos[j_1\lambda' + j_2\lambda + j_3\varpi' + j_4\varpi + j_5\Omega' + j_6\Omega]. \quad (66)$$

The same restrictions on p, p', and m apply.

In reality, the expansions of $\mathcal{R}_{\mathrm{D}}$, $\mathcal{R}_{\mathrm{E}}$, and $\mathcal{R}_{\mathrm{I}}$ are infinite series. However, in practice we are only interested in terms that are appropriate for a particular problem. Thus we need to isolate the relevant terms from the expansion, ignoring all other nonrelevant terms; effectively this assumes that the remaining terms produce only short-period effects which average out to zero. This is known as the averaging principle and it is the basis of much analytical work in Solar System dynamics. What constitutes a relevant term is not always obvious, but some general principles apply. A study of the dynamics of an object moving close to a $p+q:p$ resonance with a perturber would require the isolation of those arguments with $j_1 = \pm(p+q)$ and $j_2 = \mp p$. For example, if we were interested in studying motion close to the 5:4 resonance we would isolate those terms associated with arguments that contained $5\lambda' - 4\lambda$ and $-5\lambda' + 4\lambda$ (i.e., $p+q = \pm 5$, $q = \pm 1$).

We now consider the procedure for determining the appropriate averaged term, $\langle\mathcal{R}\rangle$ or $\langle\mathcal{R}'\rangle$ in the disturbing function, based on the fourth-order expansion given in the Appendix, or the eighth order expansion given by Murray and Harper (1993). This procedure is also given in Murray and Dermott (1999).

1. Decide which particular combination of angles, $\phi = j_1\lambda' + j_2\lambda + j_3\varpi' + j_4\varpi + j_5\Omega' + j_6\Omega$, is applicable to the problem at hand; this requires knowledge of the physical problem under investigation.
2. Determine the order, $N = |j_1 + j_2|$, of the argument. This is just the absolute value of the sum of the coefficients of λ and λ' in ϕ.
3. By looking at the appropriate order terms in the expansion of $\mathcal{R}_{\mathrm{D}}$, determine the value of the integer j which gives agreement with the desired argument, ϕ.
4. Calculate the combination of Laplace coefficients for that value of j to give the explicit form of the term of interest, $\langle\mathcal{R}_{\mathrm{D}}\rangle$ say.
5. Decide whether an external or an internal perturbation is being considered. This is determined by the nature of the problem.
6. If the perturbation is external, then look at the appropriate order terms in the expansion of the indirect part, $\mathcal{R}_{\mathrm{E}}$, and isolate a matching argument, if it exists, and read off the corresponding indirect term $\langle\mathcal{R}_{\mathrm{E}}\rangle$.
7. If the perturbation is internal, then look at the appropriate order terms in the expansion of the indirect part, $\mathcal{R}_{\mathrm{I}}$, and isolate a matching argument, if it exists, and read off the corresponding indirect term $\langle\mathcal{R}_{\mathrm{I}}\rangle$.
8. If the perturbation is external then

$$\langle\mathcal{R}\rangle = \frac{\mu'}{a'}(\langle\mathcal{R}_{\mathrm{D}}\rangle + \alpha\langle\mathcal{R}_{\mathrm{E}}\rangle). \quad (67)$$

9. If the perturbation is internal then

$$\langle\mathcal{R}'\rangle = \frac{\mu}{a}\left(\alpha\langle\mathcal{R}_{\mathrm{D}}\rangle + \frac{1}{\alpha}\langle\mathcal{R}_{\mathrm{I}}\rangle\right). \quad (68)$$

It is important to note that other terms may also have to be considered. For example, if the 2:1 resonance is being studied with a fourth-order expansion, then the contributions from arguments containing $4\lambda' - 2\lambda$, $6\lambda' - 3\lambda$, and $8\lambda' - 4\lambda$ (and their negatives) should also be included since they are associated with resonances at almost the same location and can make significant contributions if the eccentricity is large enough. Furthermore, any expansion above the first-order should also include the secular terms which contain second- and higher-order terms in the eccentricities and inclinations. These arise from arguments of order 0 which do not contain the mean longitudes. For example, inspection of the Appendix shows that the arguments 4D0.1, 4D0.2, and 4D0.3 in Table I with $j = 0$ will give rise to arguments without the mean longitudes where the associated terms contain terms of order 2 in e, e', s, and s'; all remaining secular terms are of order 4 or higher and there are no contributions from the indirect terms.

Derivation of the disturbing function should not be considered an end in itself. The actual value of $\mathcal{R}$ is unimportant because we are interested only in the terms in its series expansion which will make significant contributions to the force acting on the perturbed body. To calculate the resulting changes in orbital elements due to particular terms we need to make use of Lagrange's planetary equations (see, for example, Brouwer and Clemence 1961).

4. EXAMPLES

In order to demonstrate the use of the fourth order expansion given in the Appendix as well as the algorithm given in Section 3, we now consider two examples. In each case we show how the averaged resonant terms can be obtained using each method. Again we stress that if the derived expressions are to be used in a study of the motion of the objects then it may be necessary to include the secular as well as the resonant terms.

4.1. The Titan–Hyperion 3:4 Resonance

The saturnian satellites Titan and Hyperion are involved in a 3:4 resonance where the librating resonant argument is

$$\phi = 4\lambda' - 3\lambda - \varpi', \tag{69}$$

where the primed quantities refer to Hyperion and the unprimed ones refer to Titan. Because the mass of Titan is so large, it is customary to neglect the perturbations of Hyperion on Titan. Note that this is a first-order resonance with an internal perturber. The eccentricity of Hyperion's orbit is $e' = 0.104$ and this value is forced on the orbit as a result of the perturbations from Titan. Thus an expansion to order 1 is unlikely to be sufficient to model the system; consequently we will derive a fourth-order expansion for this particular resonant argument.

The appropriate resonant argument from the Appendix is 4D1.2 in Table IV with $j = 4$. This gives

$$\begin{aligned}\langle\mathcal{R}_\mathrm{D}\rangle = \Big\{&\frac{1}{2}e'[7+\alpha D]b^{(3)}_{\frac{1}{2}}(\alpha) \\ &+\frac{1}{8}e^2e'[-252-20\alpha D+11\alpha^2D^2+\alpha^3D^3]b^{(3)}_{\frac{1}{2}}(\alpha) \\ &+\frac{1}{16}e'^3[-358-26\alpha D+13\alpha^2D^2+\alpha^3D^3]b^{(3)}_{\frac{1}{2}}(\alpha) \\ &-\frac{1}{4}e'(s^2+s'^2)[8\alpha+\alpha^2D]\left(b^{(2)}_{\frac{3}{2}}(\alpha)+b^{(4)}_{\frac{3}{2}}(\alpha)\right)\Big\} \\ &\times\cos[4\lambda'-3\lambda-\varpi'],\end{aligned} \tag{70}$$

where the operator D denotes $\mathrm{d}/\mathrm{d}\alpha$ and where $s = \sin\frac{1}{2}I$, $s' = \sin\frac{1}{2}I'$. There are no indirect terms for this argument to this order and so $\langle\mathcal{R}'\rangle = (\mathcal{G}m/a')\langle\mathcal{R}_\mathrm{D}\rangle$.

Now consider the derivation using the algorithm discussed in Sections 2 and 3. From the definition of $\phi = +(4\lambda' - 3\lambda - \varpi')$ and the relationships given in Eqs. (39)–(51), we have $q' = 1$, $q = 0$, $\ell_\mathrm{max} = 4$, $p_\mathrm{min} = 0$, $p'_\mathrm{min} = 0$, $s_\mathrm{min} = 0$, $i_\mathrm{max} = 1$; we can also deduce that the only permissible values of n, p, and p' are $n = 0$, $p = 0$, and $p' = 0$. Hence,

$$\begin{aligned}\langle\mathcal{R}_\mathrm{D}\rangle_+ = &\sum_{i=0}^{1}\frac{(2i)!}{i!}\frac{(-1)^i}{2^{2i+1}}\alpha^i\sum_{s=0}^{i}\frac{s!}{(2s)!}\sum_{m=0}^{s}\kappa_m\frac{(s-m)!}{(s+m)!} \\ &\times F_{s,m,0}(I)\,F_{s,m,0}(I')\sum_{l=0}^{i-s}\frac{(-1)^s2^{2s}}{(i-s-l)!l!} \\ &\times\sum_{\ell=0}^{4}\frac{(-1)^\ell}{\ell!}\sum_{k=0}^{\ell}\binom{\ell}{k}(-1)^k\alpha^\ell D^\ell b^{(j)}_{i+\frac{1}{2}}(\alpha)X_3^{i+k,3}(e) \\ &\times X_4^{-(i+k+1),3}(e')\cos[4\lambda'-3\lambda-\varpi'],\end{aligned} \tag{71}$$

where $D \equiv \mathrm{d}/\mathrm{d}\alpha$. If we consider the terms arising from taking $\phi = -(4\lambda' - 3\lambda - \varpi')$ we have $q' = -1$, $q = 0$, $\ell_\mathrm{max} = 4$, $p_\mathrm{min} = 0$, $p'_\mathrm{min} = 0$, $s_\mathrm{min} = 0$, $i_\mathrm{max} = 1$; we can also deduce that the only permissible values of n, p, and p' are $n = 0$, $p = 0$, and $p' = 0$. Hence,

$$\begin{aligned}\langle\mathcal{R}_\mathrm{D}\rangle_- = &\sum_{i=0}^{1}\frac{(2i)!}{i!}\frac{(-1)^i}{2^{2i+1}}\alpha^i\sum_{s=0}^{i}\frac{s!}{(2s)!}\sum_{m=0}^{s}\kappa_m\frac{(s-m)!}{(s+m)!} \\ &\times F_{s,m,0}(I)\,F_{s,m,0}(I')\sum_{l=0}^{i-s}\frac{(-1)^s2^{2s}}{(i-s-l)!l!} \\ &\times\sum_{\ell=0}^{4}\frac{(-1)^\ell}{\ell!}\sum_{k=0}^{\ell}\binom{\ell}{k}(-1)^k\alpha^\ell D^\ell b^{(j)}_{i+\frac{1}{2}}(\alpha)X_{-3}^{i+k,-3}(e) \\ &\times X_{-4}^{-(i+k+1),-3}(e')\cos[4\lambda'-3\lambda-\varpi'].\end{aligned} \tag{72}$$

The required inclination functions are given without approximation by $F_{0,0,0}(I) = F_{0,0,0}(I') = 1$, $F_{1,1,0}(I) = 1 - s^2$ and $F_{1,1,0}(I') = 1 - s'^2$. The required Hansen coefficients to $\mathcal{O}(e^2)$ and $\mathcal{O}(e'^3)$ are given by

$$X_3^{0,3}(e) = X_{-3}^{0,-3}(e) = 1 - 9e^2 \tag{73}$$

$$X_4^{-1,3}(e') = X_{-4}^{-1,-3}(e') = \frac{7}{2}e' - \frac{179}{8}e'^3 \tag{74}$$

$$X_3^{1,3}(e) = X_{-3}^{1,-3}(e) = 1 - \frac{17}{2}e^2 \tag{75}$$

$$X_4^{-2,3}(e') = X_{-4}^{-2,-3}(e') = 4e' - 24e'^3 \tag{76}$$

$$X_3^{2,3}(e) = X_{-3}^{2,-3}(e) = 1 - \frac{15}{2}e^2 \tag{77}$$

$$X_4^{-3,3}(e') = X_{-4}^{-3,-3}(e') = \frac{9}{2}e' - 24e'^3 \tag{78}$$

$$X_3^{3,3}(e) = X_{-3}^{3,-3}(e) = 1 - 6e^2 \tag{79}$$

$$X_4^{-4,3}(e') = X_{-4}^{-4,-3}(e') = 5e' - 22e'^3 \tag{80}$$

$$X_3^{4,3}(e) = X_{-3}^{4,-3}(e) = 1 - 4e^2 \tag{81}$$

$$X_4^{-5,3}(e') = X_{-4}^{-5,-3}(e') = \frac{11}{2}e' - \frac{141}{8}e'^3 \tag{82}$$

$$X_3^{5,3}(e) = X_{-3}^{5,-3}(e) = 1 - \frac{3}{2}e^2 \tag{83}$$

$$X_4^{-6,3}(e') = X_{-4}^{-6,-3}(e') = 6e' - \frac{21}{2}e'^3. \tag{84}$$

There are no indirect terms in the expansion associated with this argument and so $\langle\mathcal{R}'\rangle = (\mathcal{G}m/a')(\langle\mathcal{R}_{\mathrm{D}}\rangle_+ + \langle\mathcal{R}_{\mathrm{D}}\rangle_-)$. The resulting expansion agrees with that obtained above using the Appendix.

A good orbital theory of Hyperion is notoriously difficult to achieve (see, for example, the study by Message 1993). It is beyond the scope of this paper to investigate this further and we content ourselves with the derivation of some of the higher-order terms which would be appropriate to include in any analytical investigation of the perturbations on Hyperion's orbit.

4.2. *The Jupiter–Pallas 18:7 Resonance*

As an example of a higher order resonance argument for which there is no existing, general literal expansion, we consider one of the terms relevant to a description of the motion of minor planet 2 Pallas. If n' and n denote the mean motions of Jupiter and Pallas, respectively, then from observation,

$$18n' - 7n = -0.45^\circ \text{ year}^{-1}. \tag{85}$$

This implies that Jupiter and Pallas are close to an 18:7 resonance. To 11th order there are 182 arguments associated with this resonance. In order to illustrate the use of our algorithm we derive the terms associated with just one of these arguments, namely

$$\phi = 18\lambda' - 7\lambda - 5\varpi - 6\Omega. \tag{86}$$

Applying the definitions given in Eqs. (39)–(45) gives $q = -5$, $q' = 0$, $\ell_{\max} = 5$, $p_{\min} = 3$, $p'_{\min} = 0$, $s_{\min} = 3$, $i_{\max} = 3$. Since $s_{\min} = i_{\max}$ the only contribution will come from $i = s = 3$ and hence $l = 0$. Similarly, since $n_{\max} = [(s-3)/2] = 0$ we must have $n = 0$. Hence, from Eq. (49) the only valid value of p is $p = 3$; hence $m = 3$ and so from Eq. (50) $p' = 0$; we also have $j = 15$. We can now write the simplified form of Eq. (52) as

$$\begin{aligned}\langle\mathcal{R}_{\mathrm{D}}\rangle_+ = {} & \frac{\alpha^3}{720}\sum_{\ell=0}^{5}\frac{(-1)^\ell}{\ell!}\sum_{k=0}^{\ell}\binom{\ell}{k}(-1)^k\alpha^\ell D^\ell b_{7/2}^{(15)}(\alpha)F_{3,3,3}(I)\\ & \times F_{3,3,0}(I')X_7^{3+k,12}(e)\,X_{18}^{-(4+k),18}(e')\\ & \times \cos[18\lambda' - 7\lambda - 5\varpi - 6\Omega].\end{aligned} \tag{87}$$

In order to complete the calculation we need to investigate the possibility that there are terms associated with the negative of our original argument, namely $\phi = -(18\lambda' - 7\lambda - 5\varpi - 6\Omega)$. In this case inspection of Eqs. (39)–(51) shows that there are no contributions and $\langle\mathcal{R}_{\mathrm{D}}\rangle_- = 0$.

We require only two evaluations of the inclination function and 12 evaluations of Hansen coefficients. Although our expansion is to 11th order, according to the approximations given in Eqs. (34)–(37) the function $F_{3,3,3}(I)$ will produce terms of $\mathcal{O}(I^6)$ and $X_7^{3+k,12}(e)$ will produce terms of $\mathcal{O}(e^5)$. Thus we are concerned only with the lowest order terms in all function evaluations. This means that we can ignore the higher order terms in $F_{3,3,0}(I') = 15 + \mathcal{O}(I'^2)$ and $X_{18}^{-(4+k),18}(e') = 1 + \mathcal{O}(e'^2)$ for $k = 0, 1, \ldots 5$. We have

$$F_{3,3,3}(I) = 15s^6 \tag{88}$$

$$X_7^{3,12}(e) = -\frac{1577149}{1280}e^5 \tag{89}$$

$$X_7^{4,12}(e) = -\frac{1473703}{960}e^5 \tag{90}$$

$$X_7^{5,12}(e) = -\frac{7280077}{3840}e^5 \tag{91}$$

$$X_7^{6,12}(e) = -\frac{1486337}{640}e^5 \tag{92}$$

$$X_7^{7,12}(e) = -\frac{10842187}{3840}e^5 \tag{93}$$

$$X_7^{8,12}(e) = -\frac{409031}{120}e^5, \tag{94}$$

and the resulting expression for $\langle\mathcal{R}_{\mathrm{D}}\rangle$ is

$$\begin{aligned}\langle\mathcal{R}_{\mathrm{D}}\rangle = {} & -\frac{e^5 s^6}{12288}[4731447\alpha^3 + 1163365\alpha^4 D + 110950\alpha^5 D^2\\ & + 5130\alpha^6 D^3 + 115\alpha^7 D^4 + \alpha^8 D^5]b_{7/2}^{(15)}(\alpha)\\ & \times\cos[18\lambda' - 7\lambda - 5\varpi - 6\Omega].\end{aligned} \tag{95}$$

Application of the algorithm given in Sections 2 and 3 for the indirect parts shows that none exist in this case; furthermore, there are no indirect terms associated with any of the 182 possible arguments to 11th order at this resonance. Therefore the averaged part of the disturbing function for this argument is given by $\langle\mathcal{R}\rangle = (\mathcal{G}m'/a')\langle\mathcal{R}_{\mathrm{D}}\rangle$. Note that this example was chosen to illustrate the calculation of a high-order term; we do not imply that this particular argument is likely to be the dominant one at the 18:7 resonance.

5. DISCUSSION

Our overriding aim in this work is to present a new, workable algorithm for producing expansions of the planetary disturbing function to any order in the eccentricities and inclinations of both objects. This obviates the need for the major expansions of the last century which, although they are still in use, have a number of drawbacks and contain errors. In particular we have now developed the capability to produce a high-order expansion for only those arguments of interest in a particular problem in Solar System dynamics. It is our hope that this work, and the fourth-order expansion given in the Appendix, will provide Solar System dynamicists with a clearer understanding of the disturbing function, its use, and its applications.

We do not pretend that the algebraic effort involved in generating and manipulating such expansions is trivial. However, an underlying assumption of this work is that software packages for algebraic manipulation will become even more widespread, powerful, and affordable than they already are. For our own purposes we have developed a package in *Mathematica* (Wolfram 1991), which implements the algorithm given here and contains a number of useful routines for generating and manipulating the series involved. The package is available free of charge to interested researchers and further information can be obtained from the second author.

APPENDIX

In Section 2 we outlined an algorithm for obtaining an expansion of the disturbing function, $\mathcal{R}$. Here we give a literal expansion of the direct part, $\mathcal{R}_{\mathrm{D}}$, and the indirect part for an external perturber, $\mathcal{R}_{\mathrm{E}}$, and an internal perturber, $\mathcal{R}_{\mathrm{I}}$, complete to fourth-order in the eccentricities and inclinations of the two bodies. The notation is the same as that used in Sections 2 and 3 where the expansion is given in terms of Laplace coefficients and their derivatives. The use of the disturbing function is described in the main text of this paper.

In our expansion each cosine argument has been labeled for the order of the expansion (4 in this case) followed by a letter denoting that the term is associated with the direct (prefix D) or indirect (prefix E or I) part of the disturbing function. The next character denotes the order of the argument, i.e., the absolute value of the sum of the coefficients of λ' and λ, and hence the order of the resonance associated with that argument. The final number identifies the argument. These are ordered with priority being given to those involving only ϖ, ϖ', Ω, and Ω' in that order. This means that terms involving e occur before those involving e', etc. This follows the procedure adopted by Murray and Harper (1993). For example, 4D3.4 denotes the fourth possible argument of the third-order direct part of the fourth order expansion. The entry for this argument should be interpreted as

$$\langle\mathcal{R}_{\mathrm{D}}\rangle = e'^3\frac{1}{48}[-6+29j-30j^2+8j^3+6\alpha D-21j\alpha D + 12j^2\alpha D - 3\alpha^2 D^2 + 6j\alpha^2 D^2 + \alpha^3 D^3]b_{\frac{1}{2}}^{(j-3)}(\alpha) \times \cos[j\lambda' + (3-j)\lambda - 3\varpi'], \quad (96)$$

where D denotes the differential operator $\mathrm{d}/\mathrm{d}\alpha$. If we consider the same argument for a particular value of j, then we must also look at the expansion of the indirect part for any matching arguments. For example, if $j=4$ and we are dealing with an external perturber then we must also include the argument 4E3.7. The total contribution of this argument to the averaged part of the disturbing function is then

$$\langle\mathcal{R}\rangle = \frac{\mathcal{G}m'}{a'}e'^3\left[-\frac{16}{3}\alpha + \frac{71}{24}b_{\frac{1}{2}}^{(1)}(\alpha) + \frac{19}{8}\alpha\frac{\mathrm{d}}{\mathrm{d}\alpha}b_{\frac{1}{2}}^{(1)}(\alpha) + \frac{7}{16}\alpha^2\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}b_{\frac{1}{2}}^{(1)}(\alpha) + \frac{1}{48}\alpha^3\frac{\mathrm{d}^3}{\mathrm{d}\alpha^3}b_{\frac{1}{2}}^{(1)}(\alpha)\right]\cos[4\lambda' - \lambda - 3\varpi']. \quad (97)$$

Tables I–XIX below give all arguments and associated terms in an expansion of the disturbing function complete to fourth-order. The expansion is arranged

TABLE I
Zeroth-Order Arguments: Direct Part

ID	Cosine argument	Term
4D0.1	$j\lambda' - j\lambda$	$f_1 + (e^2+e'^2)f_2 + (s^2+s'^2)f_3 + e^4 f_4 + e^2e'^2 f_5 + e'^4 f_6 + (e^2s^2 + e'^2s^2 + e^2s'^2 + e'^2s'^2) f_7 + (s^4+s'^4) f_8 + s^2s'^2 f_9$
4D0.2	$j\lambda' - j\lambda + \varpi' - \varpi$	$ee' f_{10} + e^3e' f_{11} + ee'^3 f_{12} + ee'(s^2+s'^2) f_{13}$
4D0.3	$j\lambda' - j\lambda + \Omega' - \Omega$	$ss' f_{14} + ss'(e^2+e'^2) f_{15} + ss'(s^2+s'^2) f_{16}$
4D0.4	$j\lambda' - j\lambda + 2\varpi' - 2\varpi$	$e^2e'^2 f_{17}$
4D0.5	$j\lambda' - j\lambda + 2\varpi - 2\Omega$	$e^2s^2 f_{18}$
4D0.6	$j\lambda' - j\lambda + \varpi' + \varpi - 2\Omega$	$ee's^2 f_{19}$
4D0.7	$j\lambda' - j\lambda + 2\varpi' - 2\Omega$	$e'^2s^2 f_{20}$
4D0.8	$j\lambda' - j\lambda + 2\varpi - \Omega' - \Omega$	$e^2ss' f_{21}$
4D0.9	$j\lambda' - j\lambda + \varpi' - \varpi - \Omega' + \Omega$	$ee'ss' f_{22}$
4D0.10	$j\lambda' - j\lambda + \varpi' - \varpi + \Omega' - \Omega$	$ee'ss' f_{23}$
4D0.11	$j\lambda' - j\lambda + \varpi' + \varpi - \Omega' - \Omega$	$ee'ss' f_{24}$
4D0.12	$j\lambda' - j\lambda + 2\varpi' - \Omega' - \Omega$	$e'^2ss' f_{25}$
4D0.13	$j\lambda' - j\lambda + 2\varpi - 2\Omega'$	$e^2s'^2 f_{18}$
4D0.14	$j\lambda' - j\lambda + \varpi' + \varpi - 2\Omega'$	$ee's'^2 f_{19}$
4D0.15	$j\lambda' - j\lambda + 2\varpi' - 2\Omega'$	$e'^2s'^2 f_{20}$
4D0.16	$j\lambda' - j\lambda + 2\Omega' - 2\Omega$	$s^2s'^2 f_{26}$

TABLE II
Zeroth-Order Arguments: Indirect Part (External and Internal Perturbers)

ID	Cosine argument	Term
4E0.1, 4I0.1	$\lambda' - \lambda$	$-1 + \frac{1}{2}(e^2+e'^2) + \frac{1}{64}(e^4+e'^4) - \frac{1}{4}e^2e'^2 + s^2 - \frac{1}{2}(e^2+e'^2) \times (s^2+s'^2) + s'^2 - s^2s'^2$
4E0.2, 4I0.2	$2\lambda' - 2\lambda - \varpi' + \varpi$	$-ee' + \frac{3}{4}e^3e' + \frac{3}{4}ee'^3 + ee's^2 + ee's'^2$
4E0.3, 4I0.3	$\lambda' - \lambda - \Omega' + \Omega$	$-2ss' + e^2ss' + e'^2ss' + s^3s' + ss'^3$
4E0.4, 4I0.4	$\lambda' - \lambda - 2\varpi' + 2\varpi$	$-\frac{1}{64}e^2e'^2$
4E0.5, 4I0.5	$3\lambda' - 3\lambda - 2\varpi' + 2\varpi$	$-\frac{81}{64}e^2e'^2$
4E0.6, 4I0.6	$\lambda' - \lambda + 2\varpi - 2\Omega$	$-\frac{1}{8}e^2s^2$
4E0.7, 4I0.7	$\lambda' - \lambda - 2\varpi' + 2\Omega$	$-\frac{1}{8}e'^2s^2$
4E0.8, 4I0.8	$\lambda' - \lambda + 2\varpi - \Omega' - \Omega$	$\frac{1}{4}e^2ss'$
4E0.9, 4I0.9	$2\lambda' - 2\lambda - \varpi' + \varpi - \Omega' + \Omega$	$-2ee'ss'$
4E0.10, 4I0.10	$\lambda' - \lambda - 2\varpi' + \Omega' + \Omega$	$\frac{1}{4}e'^2ss'$
4E0.11, 4I0.11	$\lambda' - \lambda + 2\varpi - 2\Omega'$	$-\frac{1}{8}e^2s'^2$
4E0.12, 4I0.12	$\lambda' - \lambda - 2\varpi' + 2\Omega'$	$-\frac{1}{8}e'^2s'^2$
4E0.13, 4I0.13	$\lambda' - \lambda - 2\Omega' + 2\Omega$	$-s^2s'^2$

TABLE III
Zeroth-Order Arguments: Functions of Semimajor Axis

i	f_i
1	$\frac{1}{2}A_j$
2	$\frac{1}{8}[-4j^2+2\alpha D+\alpha^2D^2]A_j$
3	$\frac{1}{4}[-\alpha]B_{j-1}+\frac{1}{4}[-\alpha]B_{j+1}$
4	$\frac{1}{128}[-9j^2+16j^4-8j^2\alpha D-8j^2\alpha^2D^2+4\alpha^3D^3+\alpha^4D^4]A_j$
5	$\frac{1}{32}[16j^4+4\alpha D-16j^2\alpha D+14\alpha^2D^2-8j^2\alpha^2D^2$ $+8\alpha^3D^3+\alpha^4D^4]A_j$
6	$\frac{1}{128}[-17j^2+16j^4+24\alpha D-24j^2\alpha D+36\alpha^2D^2-8j^2\alpha^2D^2$ $+12\alpha^3D^3+\alpha^4D^4]A_j$
7	$\frac{1}{16}[-2\alpha+4j^2\alpha-4\alpha^2D-\alpha^3D^2](B_{j-1}+B_{j+1})$
8	$\frac{3}{16}[\alpha^2]C_{j-2}+\frac{3}{4}[\alpha^2]C_j+\frac{3}{16}[\alpha^2]C_{j+2}$
9	$\frac{1}{4}[\alpha](B_{j-1}+B_{j+1})+\frac{3}{8}[\alpha^2]C_{j-2}+\frac{15}{4}[\alpha^2]C_j+\frac{3}{8}[\alpha^2]C_{j+2}$
10	$\frac{1}{4}[2+6j+4j^2-2\alpha D-\alpha^2D^2]A_{j+1}$
11	$\frac{1}{32}[-6j-26j^2-36j^3-16j^4+6j\alpha D+12j^2\alpha D-4\alpha^2D^2$ $+7j\alpha^2D^2+8j^2\alpha^2D^2-6\alpha^3D^3-\alpha^4D^4]A_{j+1}$
12	$\frac{1}{32}[4+2j-22j^2-36j^3-16j^4-4\alpha D+22j\alpha D+20j^2\alpha D$ $-22\alpha^2D^2+7j\alpha^2D^2+8j^2\alpha^2D^2-10\alpha^3D^3-\alpha^4D^4]A_{j+1}$
13	$\frac{1}{8}[-6j\alpha-4j^2\alpha+4\alpha^2D+\alpha^3D^2](B_j+B_{j+2})$
14	$[\alpha]B_{j+1}$
15	$\frac{1}{4}[2\alpha-4j^2\alpha+4\alpha^2D+\alpha^3D^2]B_{j+1}$
16	$\frac{1}{2}[-\alpha]B_{j+1}+3[-\alpha^2]C_j+\frac{3}{2}[-\alpha^2]C_{j+2}$
17	$\frac{1}{64}[12+64j+109j^2+72j^3+16j^4-12\alpha D-28j\alpha D-16j^2\alpha D$ $+6\alpha^2D^2-14j\alpha^2D^2-8j^2\alpha^2D^2+8\alpha^3D^3+\alpha^4D^4]A_{j+2}$
18	$\frac{1}{16}[12\alpha-15j\alpha+4j^2\alpha+8\alpha^2D-4j\alpha^2D+\alpha^3D^2]B_{j-1}$
19	$\frac{1}{8}[6j\alpha-4j^2\alpha-4\alpha^2D+4j\alpha^2D-\alpha^3D^2]B_j$
20	$\frac{1}{16}[3j\alpha+4j^2\alpha-4j\alpha^2D+\alpha^3D^2]B_{j+1}$
21	$\frac{1}{8}[-12\alpha+15j\alpha-4j^2\alpha-8\alpha^2D+4j\alpha^2D-\alpha^3D^2]B_{j-1}$
22	$\frac{1}{4}[6j\alpha+4j^2\alpha-4\alpha^2D-\alpha^3D^2]B_j$
23	$\frac{1}{4}[6j\alpha+4j^2\alpha-4\alpha^2D-\alpha^3D^2]B_{j+2}$
24	$\frac{1}{4}[-6j\alpha+4j^2\alpha+4\alpha^2D-4j\alpha^2D+\alpha^3D^2]B_j$
25	$\frac{1}{8}[-3j\alpha-4j^2\alpha+4j\alpha^2D-\alpha^3D^2]B_{j+1}$
26	$\frac{1}{2}[\alpha]B_{j+1}+\frac{3}{4}[\alpha^2]C_j+\frac{3}{2}[\alpha^2]C_{j+2}$

TABLE IV
First-Order Arguments: Direct Part

ID	Cosine argument	Term
4D1.1	$j\lambda'+(1-j)\lambda-\varpi$	$ef_{27}+e^3f_{28}+ee'^2f_{29}$ $+e(s^2+s'^2)f_{30}$
4D1.2	$j\lambda'+(1-j)\lambda-\varpi'$	$e'f_{31}+e^2e'f_{32}+e'^3f_{33}$ $+e'(s^2+s'^2)f_{34}$
4D1.3	$j\lambda'+(1-j)\lambda+\varpi'-2\varpi$	$e^2e'f_{35}$
4D1.4	$j\lambda'+(1-j)\lambda-2\varpi'+\varpi$	ee'^2f_{36}
4D1.5	$j\lambda'+(1-j)\lambda+\varpi-2\Omega$	es^2f_{37}
4D1.6	$j\lambda'+(1-j)\lambda+\varpi'-2\Omega$	$e's^2f_{38}$

TABLE IV—*Continued*

ID	Cosine argument	Term
4D1.7	$j\lambda'+(1-j)\lambda-\varpi-\Omega'+\Omega$	$ess'f_{39}$
4D1.8	$j\lambda'+(1-j)\lambda-\varpi+\Omega'-\Omega$	$ess'f_{40}$
4D1.9	$j\lambda'+(1-j)\lambda+\varpi-\Omega'-\Omega$	$ess'f_{41}$
4D1.10	$j\lambda'+(1-j)\lambda-\varpi'-\Omega'+\Omega$	$e'ss'f_{42}$
4D1.11	$j\lambda'+(1-j)\lambda-\varpi'+\Omega'-\Omega$	$e'ss'f_{43}$
4D1.12	$j\lambda'+(1-j)\lambda+\varpi'-\Omega'-\Omega$	$e'ss'f_{44}$
4D1.13	$j\lambda'+(1-j)\lambda+\varpi-2\Omega'$	es'^2f_{37}
4D1.14	$j\lambda'+(1-j)\lambda+\varpi'-2\Omega'$	$e's'^2f_{38}$

TABLE V
First-Order Arguments: Indirect Part (External Perturber)

ID	Cosine argument	Term
4E1.1	$\lambda'-2\lambda+\varpi$	$-\frac{1}{2}e+\frac{3}{8}e^3+\frac{1}{4}ee'^2+\frac{1}{2}es^2+\frac{1}{2}es'^2$
4E1.2	$\lambda'-\varpi$	$\frac{3}{2}e-\frac{3}{4}ee'^2-\frac{3}{2}es^2-\frac{3}{2}es'^2$
4E1.3	$2\lambda'-\lambda-\varpi'$	$-2e'+e^2e'+\frac{3}{2}e'^3+2e's^2+2e's'^2$
4E1.4	$2\lambda'-3\lambda-\varpi'+2\varpi$	$-\frac{3}{4}e^2e'$
4E1.5	$\lambda'-2\varpi'+\varpi$	$\frac{3}{16}ee'^2$
4E1.6	$3\lambda'-2\lambda-2\varpi'+\varpi$	$-\frac{27}{16}ee'^2$
4E1.7	$\lambda'+\varpi-2\Omega$	$\frac{3}{2}es^2$
4E1.8	$\lambda'-2\lambda+\varpi-\Omega'+\Omega$	$-ess'$
4E1.9	$\lambda'-\varpi-\Omega'+\Omega$	$3ess'$
4E1.10	$\lambda'+\varpi-\Omega'-\Omega$	$-3ess'$
4E1.11	$2\lambda'-\lambda-\varpi'-\Omega'+\Omega$	$-4e'ss'$
4E1.12	$\lambda'+\varpi-2\Omega'$	$\frac{3}{2}es'^2$

TABLE VI
First-Order Arguments: Indirect Part (Internal Perturber)

ID	Cosine argument	Term
4I1.1	$\lambda'-2\lambda+\varpi$	$-2e+\frac{3}{2}e^3+ee'^2+2es^2+2es'^2$
4I1.2	$\lambda-\varpi'$	$\frac{3}{2}e'-\frac{3}{4}e^2e'-\frac{3}{2}e's^2-\frac{3}{2}e's'^2$
4I1.3	$2\lambda'-\lambda-\varpi'$	$-\frac{1}{2}e'+\frac{1}{4}e^2e'+\frac{3}{8}e'^3+\frac{1}{2}e's^2+\frac{1}{2}e's'^2$
4I1.4	$\lambda+\varpi'-2\varpi$	$\frac{3}{16}e^2e'$
4I1.5	$2\lambda'-3\lambda-\varpi'+2\varpi$	$-\frac{27}{16}e^2e'$
4I1.6	$3\lambda'-2\lambda-2\varpi'+\varpi$	$-\frac{3}{4}ee'^2$
4I1.7	$\lambda+\varpi'-2\Omega$	$\frac{3}{2}e's^2$
4I1.8	$\lambda'-2\lambda+\varpi-\Omega'+\Omega$	$-4ess'$
4I1.9	$\lambda-\varpi'+\Omega'-\Omega$	$3e'ss'$
4I1.10	$\lambda+\varpi'-\Omega'-\Omega$	$-3e'ss'$
4I1.11	$2\lambda'-\lambda-\varpi'-\Omega'+\Omega$	$-e'ss'$
4I1.12	$\lambda+\varpi'-2\Omega'$	$\frac{3}{2}e's'^2$

TABLE VII
First-Order Arguments: Functions of Semimajor Axis

i	f_i
27	$\frac{1}{2}[-2j-\alpha D]A_j$
28	$\frac{1}{16}[2j-10j^2+8j^3+3\alpha D-7j\alpha D+4j^2\alpha D-2\alpha^2D^2$ $-2j\alpha^2D^2-\alpha^3D^3]A_j$
29	$\frac{1}{8}[8j^3-2\alpha D-4j\alpha D+4j^2\alpha D-4\alpha^2D^2-2j\alpha^2D^2-\alpha^3D^3]A_j$
30	$\frac{1}{4}[\alpha+2j\alpha+\alpha^2D](B_{j-1}+B_{j+1})$
31	$\frac{1}{2}[-1+2j+\alpha D]A_{j-1}$
32	$\frac{1}{8}[4-16j+20j^2-8j^3-4\alpha D+12j\alpha D-4j^2\alpha D+3\alpha^2D^2$ $+2j\alpha^2D^2+\alpha^3D^3]A_{j-1}$
33	$\frac{1}{16}[-2-j+10j^2-8j^3+2\alpha D+9j\alpha D-4j^2\alpha D+5\alpha^2D^2$ $+2j\alpha^2D^2+\alpha^3D^3]A_{j-1}$
34	$\frac{1}{4}[-2j\alpha-\alpha^2D](B_{j-2}+B_j)$
35	$\frac{1}{16}[1-j-10j^2-8j^3-\alpha D-j\alpha D-4j^2\alpha D+3\alpha^2D^2$ $+2j\alpha^2D^2-\alpha^3D^3]A_{j+1}$
36	$\frac{1}{16}[-8+32j-30j^2+8j^3+8\alpha D-17j\alpha D+4j^2\alpha D-4\alpha^2D^2$ $-2j\alpha^2D^2-\alpha^3D^3]A_{j-2}$
37	$\frac{1}{4}[-5\alpha+2j\alpha-\alpha^2D]B_{j-1}$
38	$\frac{1}{4}[-2j\alpha+\alpha^2D]B_j$
39	$\frac{1}{2}[-\alpha-2j\alpha-\alpha^2D]B_{j-1}$
40	$\frac{1}{2}[-\alpha-2j\alpha-\alpha^2D]B_{j+1}$
41	$\frac{1}{2}[5\alpha-2j\alpha+\alpha^2D]B_{j-1}$
42	$\frac{1}{2}[2j\alpha+\alpha^2D]B_{j-2}$
43	$\frac{1}{2}[2j\alpha+\alpha^2D]B_j$
44	$\frac{1}{2}[2j\alpha-\alpha^2D]B_j$

TABLE VIII
Second-Order Arguments: Direct Part

ID	Cosine argument	Term
4D2.1	$j\lambda'+(2-j)\lambda-2\varpi$	$e^2f_{45}+e^4f_{46}+e^2e'^2f_{47}$ $+e^2(s^2+s'^2)f_{48}$
4D2.2	$j\lambda'+(2-j)\lambda-\varpi'-\varpi$	$ee'f_{49}+e^3e'f_{50}+ee'^3f_{51}$ $+ee'(s^2+s'^2)f_{52}$
4D2.3	$j\lambda'+(2-j)\lambda-2\varpi'$	$e'^2f_{53}+e^2e'^2f_{54}+e'^4f_{55}$ $+e'^2(s^2+s'^2)f_{56}$
4D2.4	$j\lambda'+(2-j)\lambda-2\Omega$	$s^2f_{57}+e^2s^2f_{58}+e'^2s^2f_{59}$ $+s^4f_{60}+s^2s'^2f_{61}$
4D2.5	$j\lambda'+(2-j)\lambda-\Omega'-\Omega$	$ss'f_{62}+e^2ss'f_{63}+e'^2ss'f_{64}$ $+s^3s'f_{65}+ss'^3f_{66}$
4D2.6	$j\lambda'+(2-j)\lambda-2\Omega'$	$s'^2f_{57}+e^2s'^2f_{58}+e'^2s'^2f_{59}$ $+s^2s'^2f_{67}+s'^4f_{60}$

TABLE VIII—*Continued*

ID	Cosine argument	Term
4D2.7	$j\lambda'+(2-j)\lambda+\varpi'-3\varpi$	$e^3e'f_{68}$
4D2.8	$j\lambda'+(2-j)\lambda-3\varpi'+\varpi$	ee'^3f_{69}
4D2.9	$j\lambda'+(2-j)\lambda-\varpi'+\varpi-2\Omega$	$ee's^2f_{70}$
4D2.10	$j\lambda'+(2-j)\lambda+\varpi'-\varpi-2\Omega$	$ee's^2f_{71}$
4D2.11	$j\lambda'+(2-j)\lambda-2\varpi-\Omega'+\Omega$	$e^2ss'f_{72}$
4D2.12	$j\lambda'+(2-j)\lambda-2\varpi+\Omega'-\Omega$	$e^2ss'f_{73}$
4D2.13	$j\lambda'+(2-j)\lambda-\varpi'-\varpi-\Omega'+\Omega$	$ee'ss'f_{74}$
4D2.14	$j\lambda'+(2-j)\lambda-\varpi'-\varpi+\Omega'-\Omega$	$ee'ss'f_{75}$
4D2.15	$j\lambda'+(2-j)\lambda-\varpi'+\varpi-\Omega'-\Omega$	$ee'ss'f_{76}$
4D2.16	$j\lambda'+(2-j)\lambda+\varpi'-\varpi-\Omega'-\Omega$	$ee'ss'f_{77}$
4D2.17	$j\lambda'+(2-j)\lambda-2\varpi'-\Omega'+\Omega$	$e'^2ss'f_{78}$
4D2.18	$j\lambda'+(2-j)\lambda-2\varpi'+\Omega'-\Omega$	$e'^2ss'f_{79}$
4D2.19	$j\lambda'+(2-j)\lambda+\Omega'-3\Omega$	$s^3s'f_{80}$
4D2.20	$j\lambda'+(2-j)\lambda-\varpi'+\varpi-2\Omega'$	$ee's'^2f_{70}$
4D2.21	$j\lambda'+(2-j)\lambda+\varpi'-\varpi-2\Omega'$	$ee's'^2f_{71}$
4D2.22	$j\lambda'+(2-j)\lambda-3\Omega'+\Omega$	ss'^3f_{81}

TABLE IX
Second-Order Arguments: Indirect Part (External Perturber)

ID	Cosine argument	Term
4E2.1	$\lambda'-3\lambda+2\varpi$	$-\frac{3}{8}e^2+\frac{3}{8}e^4+\frac{3}{16}e^2e'^2+\frac{3}{8}e^2s^2$ $+\frac{3}{8}e^2s'^2$
4E2.2	$\lambda'+\lambda-2\varpi$	$-\frac{1}{8}e^2-\frac{1}{24}e^4+\frac{1}{16}e^2e'^2+\frac{1}{8}e^2s^2$ $+\frac{1}{8}e^2s'^2$
4E2.3	$2\lambda'-\varpi'-\varpi$	$3ee'-\frac{9}{4}ee'^3-3ee's^2-3ee's'^2$
4E2.4	$\lambda'+\lambda-2\varpi'$	$-\frac{1}{8}e'^2+\frac{1}{16}e^2e'^2-\frac{1}{24}e'^4+\frac{1}{8}e'^2s^2$ $+\frac{1}{8}e'^2s'^2$
4E2.5	$3\lambda'-\lambda-2\varpi'$	$-\frac{27}{8}e'^2+\frac{27}{16}e^2e'^2+\frac{27}{8}e'^4+\frac{27}{8}e'^2s^2$ $+\frac{27}{8}e'^2s'^2$
4E2.6	$\lambda'+\lambda-2\Omega$	$-s^2+\frac{1}{2}e^2s^2+\frac{1}{2}e'^2s^2+s^2s'^2$
4E2.7	$\lambda'+\lambda-\Omega'-\Omega$	$2ss'-e^2ss'-e'^2ss'-s^3s'-ss'^3$
4E2.8	$\lambda'+\lambda-2\Omega'$	$-s'^2+\frac{1}{2}e^2s'^2+\frac{1}{2}e'^2s'^2+s^2s'^2$
4E2.9	$2\lambda'-4\lambda-\varpi'+3\varpi$	$-\frac{2}{3}e^3e'$
4E2.10	$2\lambda'-3\varpi'+\varpi$	$\frac{1}{4}ee'^3$
4E2.11	$4\lambda'-2\lambda-3\varpi'+\varpi$	$-\frac{8}{3}ee'^3$
4E2.12	$2\lambda'-\varpi'+\varpi-2\Omega$	$3ee's^2$
4E2.13	$\lambda'-3\lambda+2\varpi-\Omega'+\Omega$	$-\frac{3}{4}e^2ss'$
4E2.14	$\lambda'+\lambda-2\varpi-\Omega'+\Omega$	$-\frac{1}{4}e^2ss'$
4E2.15	$2\lambda'-\varpi'-\varpi-\Omega'+\Omega$	$6ee'ss'$
4E2.16	$2\lambda'-\varpi'+\varpi-\Omega'-\Omega$	$-6ee'ss'$
4E2.17	$\lambda'+\lambda-2\varpi'+\Omega'-\Omega$	$-\frac{1}{4}e'^2ss'$
4E2.18	$3\lambda'-\lambda-2\varpi'-\Omega'+\Omega$	$-\frac{27}{4}e'^2ss'$
4E2.19	$2\lambda'-\varpi'+\varpi-2\Omega'$	$3ee's'^2$

TABLE X
Second-Order Arguments: Indirect Part (Internal Perturber)

ID	Cosine argument	Term
4I2.1	$\lambda' - 3\lambda + 2\varpi$	$-\frac{27}{8}e^2 + \frac{27}{8}e^4 + \frac{27}{16}e^2e'^2 + \frac{27}{8}e^2s^2 + \frac{27}{8}e^2s'^2$
4I2.2	$\lambda' + \lambda - 2\varpi$	$-\frac{1}{8}e^2 - \frac{1}{24}e^4 + \frac{1}{16}e^2e'^2 + \frac{1}{8}e^2s^2 + \frac{1}{8}e^2s'^2$
4I2.3	$2\lambda - \varpi' - \varpi$	$3ee' - \frac{9}{4}e^3e' - 3ee's^2 - 3ee's'^2$
4I2.4	$\lambda' + \lambda - 2\varpi'$	$-\frac{1}{8}e'^2 + \frac{1}{16}e^2e'^2 - \frac{1}{24}e'^4 + \frac{1}{8}e'^2s^2 + \frac{1}{8}e'^2s'^2$
4I2.5	$3\lambda' - \lambda - 2\varpi'$	$-\frac{3}{8}e'^2 + \frac{3}{16}e^2e'^2 + \frac{3}{8}e'^4 + \frac{3}{8}e'^2s^2 + \frac{3}{8}e'^2s'^2$
4I2.6	$\lambda' + \lambda - 2\Omega$	$-s^2 + \frac{1}{2}e^2s^2 + \frac{1}{2}e'^2s^2 + s^2s'^2$
4I2.7	$\lambda' + \lambda - \Omega' - \Omega$	$2ss' - e^2ss' - e'^2ss' - s^3s' - ss'^3$
4I2.8	$\lambda' + \lambda - 2\Omega'$	$-s'^2 + \frac{1}{2}e^2s'^2 + \frac{1}{2}e'^2s'^2 + s^2s'^2$
4I2.9	$2\lambda + \varpi' - 3\varpi$	$\frac{1}{4}e^3e'$
4I2.10	$2\lambda' - 4\lambda - \varpi' + 3\varpi$	$-\frac{8}{3}e^3e'$
4I2.11	$4\lambda' - 2\lambda - 3\varpi' + \varpi$	$-\frac{2}{3}ee'^3$
4I2.12	$2\lambda + \varpi' - \varpi - 2\Omega$	$3ee's^2$
4I2.13	$\lambda' - 3\lambda + 2\varpi - \Omega' + \Omega$	$-\frac{27}{4}e^2ss'$
4I2.14	$\lambda' + \lambda - 2\varpi - \Omega' + \Omega$	$-\frac{1}{4}e^2ss'$
4I2.15	$2\lambda - \varpi' - \varpi + \Omega' - \Omega$	$6ee'ss'$
4I2.16	$2\lambda + \varpi' - \varpi - \Omega' - \Omega$	$-6ee'ss'$
4I2.17	$\lambda' + \lambda - 2\varpi' + \Omega' - \Omega$	$-\frac{1}{4}e'^2ss'$
4I2.18	$3\lambda' - \lambda - 2\varpi' - \Omega' + \Omega$	$-\frac{3}{4}e'^2ss'$
4I2.19	$2\lambda + \varpi' - \varpi - 2\Omega'$	$3ee's'^2$

TABLE XI
Second-Order Arguments: Functions of Semimajor Axis

i	f_i
45	$\frac{1}{8}[-5j + 4j^2 - 2\alpha D + 4j\alpha D + \alpha^2D^2]A_j$
46	$\frac{1}{96}[22j - 64j^2 + 60j^3 - 16j^4 + 16\alpha D - 46j\alpha D + 48j^2\alpha D - 16j^3\alpha D - 12\alpha^2D^2 + 9j\alpha^2D^2 + 4j\alpha^3D^3 + \alpha^4D^4]A_j$
47	$\frac{1}{32}[20j^3 - 16j^4 - 4\alpha D - 2j\alpha D + 16j^2\alpha D - 16j^3\alpha D - 2\alpha^2D^2 + 11j\alpha^2D^2 + 4\alpha^3D^3 + 4j\alpha^3D^3 + \alpha^4D^4]A_j$
48	$\frac{1}{16}[2\alpha + j\alpha - 4j^2\alpha - 4j\alpha^2D - \alpha^3D^2](B_{j-1} + B_{j+1})$
49	$\frac{1}{4}[-2 + 6j - 4j^2 + 2\alpha D - 4j\alpha D - \alpha^2D^2]A_{j-1}$
50	$\frac{1}{32}[20 - 86j + 126j^2 - 76j^3 + 16j^4 - 20\alpha D + 74j\alpha D - 64j^2\alpha D + 16j^3\alpha D + 14\alpha^2D^2 - 17j\alpha^2D^2 - 2\alpha^3D^3 - 4j\alpha^3D^3 - \alpha^4D^4]A_{j-1}$

TABLE XI—*Continued*

i	f_i
51	$\frac{1}{32}[-4 + 2j + 22j^2 - 36j^3 + 16j^4 + 4\alpha D + 6j\alpha D - 32j^2\alpha D + 16j^3\alpha D - 2\alpha^2D^2 - 19j\alpha^2D^2 - 6\alpha^3D^3 - 4j\alpha^3D^3 - \alpha^4D^4]A_{j-1}$
52	$\frac{1}{8}[-2j\alpha + 4j^2\alpha + 4j\alpha^2D + \alpha^3D^2](B_{j-2} + B_j)$
53	$\frac{1}{8}[2 - 7j + 4j^2 - 2\alpha D + 4j\alpha D + \alpha^2D^2]A_{j-2}$
54	$\frac{1}{32}[-32 + 144j - 184j^2 + 92j^3 - 16j^4 + 32\alpha D - 102j\alpha D + 80j^2\alpha D - 16j^3\alpha D - 16\alpha^2D^2 + 25j\alpha^2D^2 + 4\alpha^3D^3 + 4j\alpha^3D^3 + \alpha^4D^4]A_{j-2}$
55	$\frac{1}{96}[12 - 14j - 40j^2 + 52j^3 - 16j^4 - 12\alpha D - 10j\alpha D + 48j^2\alpha D - 16j^3\alpha D + 6\alpha^2D^2 + 27j\alpha^2D^2 + 8\alpha^3D^3 + 4j\alpha^3D^3 + \alpha^4D^4]A_{j-2}$
56	$\frac{1}{16}[3j\alpha - 4j^2\alpha - 4j\alpha^2D - \alpha^3D^2](B_{j-3} + B_{j-1})$
57	$\frac{1}{2}[\alpha]B_{j-1}$
58	$\frac{1}{8}[-14\alpha + 16j\alpha - 4j^2\alpha + 4\alpha^2D + \alpha^3D^2]B_{j-1}$
59	$\frac{1}{8}[2\alpha - 4j^2\alpha + 4\alpha^2D + \alpha^3D^2]B_{j-1}$
60	$\frac{3}{4}[-\alpha^2]C_{j-2} + \frac{3}{4}[-\alpha^2]C_j$
61	$\frac{1}{2}[-\alpha]B_{j-1} + \frac{3}{4}[-\alpha^2]C_{j-2} + \frac{15}{4}[-\alpha^2]C_j$
62	$[-\alpha]B_{j-1}$
63	$\frac{1}{4}[14\alpha - 16j\alpha + 4j^2\alpha - 4\alpha^2D - \alpha^3D^2]B_{j-1}$
64	$\frac{1}{4}[-2\alpha + 4j^2\alpha - 4\alpha^2D - \alpha^3D^2]B_{j-1}$
65	$\frac{1}{2}[\alpha]B_{j-1} + 3[\alpha^2]C_{j-2} + \frac{3}{2}[\alpha^2]C_j$
66	$\frac{1}{2}[\alpha]B_{j-1} + \frac{3}{2}[\alpha^2]C_{j-2} + 3[\alpha^2]C_j$
67	$\frac{1}{2}[-\alpha]B_{j-1} + \frac{15}{4}[-\alpha^2]C_{j-2} + \frac{3}{4}[-\alpha^2]C_j$
68	$\frac{1}{96}[4 - 2j - 26j^2 - 4j^3 + 16j^4 - 4\alpha D - 2j\alpha D + 16j^3\alpha D + 6\alpha^2D^2 - 3j\alpha^2D^2 - 2\alpha^3D^3 - 4j\alpha^3D^3 - \alpha^4D^4]A_{j+1}$
69	$\frac{1}{96}[36 - 186j + 238j^2 - 108j^3 + 16j^4 - 36\alpha D + 130j\alpha D - 96j^2\alpha D + 16j^3\alpha D + 18\alpha^2D^2 - 33j\alpha^2D^2 - 6\alpha^3D^3 - 4j\alpha^3D^3 - \alpha^4D^4]A_{j-3}$
70	$\frac{1}{8}[-14j\alpha + 4j^2\alpha - 8\alpha^2D - \alpha^3D^2]B_{j-2}$
71	$\frac{1}{8}[-2j\alpha + 4j^2\alpha - \alpha^3D^2]B_j$
72	$\frac{1}{8}[-2\alpha - j\alpha + 4j^2\alpha + 4j\alpha^2D + \alpha^3D^2]B_{j-1}$
73	$\frac{1}{8}[-2\alpha - j\alpha + 4j^2\alpha + 4j\alpha^2D + \alpha^3D^2]B_{j+1}$
74	$\frac{1}{4}[2j\alpha - 4j^2\alpha - 4j\alpha^2D - \alpha^3D^2]B_{j-2}$
75	$\frac{1}{4}[2j\alpha - 4j^2\alpha - 4j\alpha^2D - \alpha^3D^2]B_j$
76	$\frac{1}{4}[14j\alpha - 4j^2\alpha + 8\alpha^2D + \alpha^3D^2]B_{j-2}$
77	$\frac{1}{4}[2j\alpha - 4j^2\alpha + \alpha^3D^2]B_j$
78	$\frac{1}{8}[-3j\alpha + 4j^2\alpha + 4j\alpha^2D + \alpha^3D^2]B_{j-3}$
79	$\frac{1}{8}[-3j\alpha + 4j^2\alpha + 4j\alpha^2D + \alpha^3D^2]B_{j-1}$
80	$\frac{3}{2}[\alpha^2]C_j$
81	$\frac{3}{2}[\alpha^2]C_{j-2}$

TABLE XII
Third-Order Arguments: Direct Part

ID	Cosine argument	Term
4D3.1	$j\lambda' + (3-j)\lambda - 3\varpi$	$e^3 f_{82}$
4D3.2	$j\lambda' + (3-j)\lambda - \varpi' - 2\varpi$	$e^2e' f_{83}$
4D3.3	$j\lambda' + (3-j)\lambda - 2\varpi' - \varpi$	$ee'^2 f_{84}$
4D3.4	$j\lambda' + (3-j)\lambda - 3\varpi'$	$e'^3 f_{85}$
4D3.5	$j\lambda' + (3-j)\lambda - \varpi - 2\Omega$	$es^2 f_{86}$
4D3.6	$j\lambda' + (3-j)\lambda - \varpi' - 2\Omega$	$e's^2 f_{87}$
4D3.7	$j\lambda' + (3-j)\lambda - \varpi - \Omega' - \Omega$	$ess' f_{88}$
4D3.8	$j\lambda' + (3-j)\lambda - \varpi' - \Omega' - \Omega$	$e'ss' f_{89}$
4D3.9	$j\lambda' + (3-j)\lambda - \varpi - 2\Omega'$	$es'^2 f_{86}$
4D3.10	$j\lambda' + (3-j)\lambda - \varpi' - 2\Omega'$	$e's'^2 f_{87}$

TABLE XIII
Third-Order Arguments: Indirect Part (External Perturber)

ID	Cosine argument	Term
4E3.1	$\lambda' - 4\lambda + 3\varpi$	$-\frac{1}{3}e^3$
4E3.2	$\lambda' + 2\lambda - 3\varpi$	$-\frac{1}{24}e^3$
4E3.3	$2\lambda' + \lambda - \varpi' - 2\varpi$	$-\frac{1}{4}e^2e'$
4E3.4	$\lambda' + 2\lambda - 2\varpi' - \varpi$	$-\frac{1}{16}ee'^2$
4E3.5	$3\lambda' - 2\varpi' - \varpi$	$\frac{81}{16}ee'^2$
4E3.6	$2\lambda' + \lambda - 3\varpi'$	$-\frac{1}{6}e'^3$
4E3.7	$4\lambda' - \lambda - 3\varpi'$	$-\frac{16}{3}e'^3$
4E3.8	$\lambda' + 2\lambda - \varpi - 2\Omega$	$-\frac{1}{2}es^2$
4E3.9	$2\lambda' + \lambda - \varpi' - 2\Omega$	$-2e's^2$
4E3.10	$\lambda' + 2\lambda - \varpi - \Omega' - \Omega$	ess'
4E3.11	$2\lambda' + \lambda - \varpi' - \Omega' - \Omega$	$4e'ss'$
4E3.12	$\lambda' + 2\lambda - \varpi - 2\Omega'$	$-\frac{1}{2}es'^2$
4E3.13	$2\lambda' + \lambda - \varpi' - 2\Omega'$	$-2e's'^2$

TABLE XIV
Third-Order Arguments: Indirect Part (Internal Perturber)

ID	Cosine argument	Term
4I3.1	$\lambda' - 4\lambda + 3\varpi$	$-\frac{16}{3}e^3$
4I3.2	$\lambda' + 2\lambda - 3\varpi$	$-\frac{1}{6}e^3$
4I3.3	$3\lambda - \varpi' - 2\varpi$	$\frac{81}{16}e^2e'$
4I3.4	$2\lambda' + \lambda - \varpi' - 2\varpi$	$-\frac{1}{16}e^2e'$
4I3.5	$\lambda' + 2\lambda - 2\varpi' - \varpi$	$-\frac{1}{4}ee'^2$
4I3.6	$2\lambda' + \lambda - 3\varpi'$	$-\frac{1}{24}e'^3$
4I3.7	$4\lambda' - \lambda - 3\varpi'$	$-\frac{1}{3}e'^3$
4I3.8	$\lambda' + 2\lambda - \varpi - 2\Omega$	$-2es^2$
4I3.9	$2\lambda' + \lambda - \varpi' - 2\Omega$	$-\frac{1}{2}e's^2$
4I3.10	$\lambda' + 2\lambda - \varpi - \Omega' - \Omega$	$4ess'$
4I3.11	$2\lambda' + \lambda - \varpi' - \Omega' - \Omega$	$e'ss'$
4I3.12	$\lambda' + 2\lambda - \varpi - 2\Omega'$	$-2es'^2$
4I3.13	$2\lambda' + \lambda - \varpi' - 2\Omega'$	$-\frac{1}{2}e's'^2$

TABLE XV
Third-Order Arguments: Functions of Semimajor Axis

i	f_i
82	$\frac{1}{48}[-26j + 30j^2 - 8j^3 - 9\alpha D + 27j\alpha D - 12j^2\alpha D + 6\alpha^2 D^2 - 6j\alpha^2 D^2 - \alpha^3 D^3]A_j$
83	$\frac{1}{16}[-9 + 31j - 30j^2 + 8j^3 + 9\alpha D - 25j\alpha D + 12j^2\alpha D - 5\alpha^2 D^2 + 6j\alpha^2 D^2 + \alpha^3 D^3]A_{j-1}$
84	$\frac{1}{16}[8 - 32j + 30j^2 - 8j^3 - 8\alpha D + 23j\alpha D - 12j^2\alpha D + 4\alpha^2 D^2 - 6j\alpha^2 D^2 - \alpha^3 D^3]A_{j-2}$
85	$\frac{1}{48}[-6 + 29j - 30j^2 + 8j^3 + 6\alpha D - 21j\alpha D + 12j^2\alpha D - 3\alpha^2 D^2 + 6j\alpha^2 D^2 + \alpha^3 D^3]A_{j-3}$
86	$\frac{1}{4}[3\alpha - 2j\alpha - \alpha^2 D]B_{j-1}$
87	$\frac{1}{4}[2j\alpha + \alpha^2 D]B_{j-2}$
88	$\frac{1}{2}[-3\alpha + 2j\alpha + \alpha^2 D]B_{j-1}$
89	$\frac{1}{2}[-2j\alpha - \alpha^2 D]B_{j-2}$

by the order of the argument, i.e., the absolute value of the sum of the coefficients of the mean longitudes in each argument. Tables I–III, Tables IV–VII, Tables VIII–XI, Tables XII–XV, and Tables XVI–XIX contain the arguments and terms for the zeroth-, first-, second-, third-, and fourth-order arguments, respectively. The full, fourth-order expansion of $\mathcal{R}$ (or $\mathcal{R}'$) should be considered as the sum of the direct and indirect terms for an external (or internal) perturber over all values of the integer j.

TABLE XVI
Fourth-Order Arguments: Direct Part

ID	Cosine argument	Term
4D4.1	$j\lambda' + (4-j)\lambda - 4\varpi$	$e^4 f_{90}$
4D4.2	$j\lambda' + (4-j)\lambda - \varpi' - 3\varpi$	$e^3e' f_{91}$
4D4.3	$j\lambda' + (4-j)\lambda - 2\varpi' - 2\varpi$	$e^2e'^2 f_{92}$
4D4.4	$j\lambda' + (4-j)\lambda - 3\varpi' - \varpi$	$ee'^3 f_{93}$
4D4.5	$j\lambda' + (4-j)\lambda - 4\varpi'$	$e'^4 f_{94}$
4D4.6	$j\lambda' + (4-j)\lambda - 2\varpi - 2\Omega$	$e^2s^2 f_{95}$
4D4.7	$j\lambda' + (4-j)\lambda - \varpi' - \varpi - 2\Omega$	$ee's^2 f_{96}$
4D4.8	$j\lambda' + (4-j)\lambda - 2\varpi' - 2\Omega$	$e'^2s^2 f_{97}$
4D4.9	$j\lambda' + (4-j)\lambda - 4\Omega$	$s^4 f_{98}$
4D4.10	$j\lambda' + (4-j)\lambda - 2\varpi - \Omega' - \Omega$	$e^2ss' f_{99}$
4D4.11	$j\lambda' + (4-j)\lambda - \varpi' - \varpi - \Omega' - \Omega$	$ee'ss' f_{100}$
4D4.12	$j\lambda' + (4-j)\lambda - 2\varpi' - \Omega' - \Omega$	$e'^2ss' f_{101}$
4D4.13	$j\lambda' + (4-j)\lambda - \Omega' - 3\Omega$	$s^3s' f_{102}$
4D4.14	$j\lambda' + (4-j)\lambda - 2\varpi - 2\Omega'$	$e^2s'^2 f_{95}$
4D4.15	$j\lambda' + (4-j)\lambda - \varpi' - \varpi - 2\Omega'$	$ee's'^2 f_{96}$
4D4.16	$j\lambda' + (4-j)\lambda - 2\varpi' - 2\Omega'$	$e'^2s'^2 f_{97}$
4D4.17	$j\lambda' + (4-j)\lambda - 2\Omega' - 2\Omega$	$s^2s'^2 f_{103}$
4D4.18	$j\lambda' + (4-j)\lambda - 3\Omega' - \Omega$	$ss'^3 f_{102}$
4D4.19	$j\lambda' + (4-j)\lambda - 4\Omega'$	$s'^4 f_{98}$

TABLE XVII
Fourth-Order Arguments: Indirect Part (External Perturber)

ID	Cosine argument	Term
4E4.1	$\lambda' - 5\lambda + 4\varpi$	$-\frac{125}{384}e^4$
4E4.2	$\lambda' + 3\lambda - 4\varpi$	$-\frac{3}{128}e^4$
4E4.3	$2\lambda' + 2\lambda - \varpi' - 3\varpi$	$-\frac{1}{12}e^3e'$
4E4.4	$\lambda' + 3\lambda - 2\varpi' - 2\varpi$	$-\frac{3}{64}e^2e'^2$
4E4.5	$3\lambda' + \lambda - 2\varpi' - 2\varpi$	$-\frac{27}{64}e^2e'^2$
4E4.6	$2\lambda' + 2\lambda - 3\varpi' - \varpi$	$-\frac{1}{12}ee'^3$
4E4.7	$4\lambda' - 3\varpi' - \varpi$	$8ee'^3$
4E4.8	$3\lambda' + \lambda - 4\varpi'$	$-\frac{27}{128}e'^4$
4E4.9	$5\lambda' - \lambda - 4\varpi'$	$-\frac{3125}{384}e'^4$
4E4.10	$\lambda' + 3\lambda - 2\varpi - 2\Omega$	$-\frac{3}{8}e^2s^2$
4E4.11	$2\lambda' + 2\lambda - \varpi' - \varpi - 2\Omega$	$-ee's^2$
4E4.12	$3\lambda' + \lambda - 2\varpi' - 2\Omega$	$-\frac{27}{8}e'^2s^2$
4E4.13	$\lambda' + 3\lambda - 2\varpi - \Omega' - \Omega$	$\frac{3}{4}e^2ss'$
4E4.14	$2\lambda' + 2\lambda - \varpi' - \varpi - \Omega' - \Omega$	$2ee'ss'$
4E4.15	$3\lambda' + \lambda - 2\varpi' - \Omega' - \Omega$	$\frac{27}{4}e'^2ss'$
4E4.16	$\lambda' + 3\lambda - 2\varpi - 2\Omega'$	$-\frac{3}{8}e^2s'^2$
4E4.17	$2\lambda' + 2\lambda - \varpi' - \varpi - 2\Omega'$	$-ee's'^2$
4E4.18	$3\lambda' + \lambda - 2\varpi' - 2\Omega'$	$-\frac{27}{8}e'^2s'^2$

Throughout the definitions of the functions of semimajor axis in Tables III, VII, XI, XV, and XIX the following notation is used:

$$A_j = b^{(j)}_{\frac{1}{2}}(\alpha) \tag{98}$$

TABLE XVIII
Fourth-Order Arguments: Indirect Part (Internal Perturber)

ID	Cosine argument	Term
4I4.1	$\lambda' - 5\lambda + 4\varpi$	$-\frac{3125}{384}e^4$
4I4.2	$\lambda' + 3\lambda - 4\varpi$	$-\frac{27}{128}e^4$
4I4.3	$4\lambda - \varpi' - 3\varpi$	$8e^3e'$
4I4.4	$2\lambda' + 2\lambda - \varpi' - 3\varpi$	$-\frac{1}{12}e^3e'$
4I4.5	$\lambda' + 3\lambda - 2\varpi' - 2\varpi$	$-\frac{27}{64}e^2e'^2$
4I4.6	$3\lambda' + \lambda - 2\varpi' - 2\varpi$	$-\frac{3}{64}e^2e'^2$
4I4.7	$2\lambda' + 2\lambda - 3\varpi' - \varpi$	$-\frac{1}{12}ee'^3$
4I4.8	$3\lambda' + \lambda - 4\varpi'$	$-\frac{3}{128}e'^4$
4I4.9	$5\lambda' - \lambda - 4\varpi'$	$-\frac{125}{384}e'^4$
4I4.10	$\lambda' + 3\lambda - 2\varpi - 2\Omega$	$-\frac{27}{8}e^2s^2$
4I4.11	$2\lambda' + 2\lambda - \varpi' - \varpi - 2\Omega$	$-ee's^2$
4I4.12	$3\lambda' + \lambda - 2\varpi' - 2\Omega$	$-\frac{3}{8}e'^2s^2$
4I4.13	$\lambda' + 3\lambda - 2\varpi - \Omega' - \Omega$	$\frac{27}{4}e^2ss'$
4I4.14	$2\lambda' + 2\lambda - \varpi' - \varpi - \Omega' - \Omega$	$2ee'ss'$
4I4.15	$3\lambda' + \lambda - 2\varpi' - \Omega' - \Omega$	$\frac{3}{4}e'^2ss'$
4I4.16	$\lambda' + 3\lambda - 2\varpi - 2\Omega'$	$-\frac{27}{8}e^2s'^2$
4I4.17	$2\lambda' + 2\lambda - \varpi' - \varpi - 2\Omega'$	$-ee's'^2$
4I4.18	$3\lambda' + \lambda - 2\varpi' - 2\Omega'$	$-\frac{3}{8}e'^2s'^2$

TABLE XIX
Fourth-Order Arguments: Functions of Semimajor Axis

i	f_i
90	$\frac{1}{384}[-206j + 283j^2 - 120j^3 + 16j^4 - 64\alpha D + 236j\alpha D - 168j^2\alpha D + 32j^3\alpha D + 48\alpha^2D^2 - 78j\alpha^2D^2 + 24j^2\alpha^2D^2 - 12\alpha^3D^3 + 8j\alpha^3D^3 + \alpha^4D^4]A_j$
91	$\frac{1}{96}[-64 + 238j - 274j^2 + 116j^3 - 16j^4 + 64\alpha D - 206j\alpha D + 156j^2\alpha D - 32j^3\alpha D - 36\alpha^2D^2 + 69j\alpha^2D^2 - 24j^2\alpha^2D^2 + 10\alpha^3D^3 - 8j\alpha^3D^3 - \alpha^4D^4]A_{j-1}$
92	$\frac{1}{64}[52 - 224j + 259j^2 - 112j^3 + 16j^4 - 52\alpha D + 176j\alpha D - 144j^2\alpha D + 32j^3\alpha D + 26\alpha^2D^2 - 60j\alpha^2D^2 + 24j^2\alpha^2D^2 - 8\alpha^3D^3 + 8j\alpha^3D^3 + \alpha^4D^4]A_{j-2}$
93	$\frac{1}{96}[-36 + 186j - 238j^2 + 108j^3 - 16j^4 + 36\alpha D - 146j\alpha D + 132j^2\alpha D - 32j^3\alpha D - 18\alpha^2D^2 + 51j\alpha^2D^2 - 24j^2\alpha^2D^2 + 6\alpha^3D^3 - 8j\alpha^3D^3 - \alpha^4D^4]A_{j-3}$
94	$\frac{1}{384}[24 - 146j + 211j^2 - 104j^3 + 16j^4 - 24\alpha D + 116j\alpha D - 120j^2\alpha D + 32j^3\alpha D + 12\alpha^2D^2 - 42j\alpha^2D^2 + 24J^2\alpha^2D^2 - 4\alpha^3D^3 + 8j\alpha^3D^3 + \alpha^4D^4]A_{j-4}$
95	$\frac{1}{16}[16\alpha - 17j\alpha + 4j^2\alpha - 8\alpha^2D + 4j\alpha^2D + \alpha^3D^2]B_{j-1}$
96	$\frac{1}{8}[10j\alpha - 4j^2\alpha + 4\alpha^2D - 4j\alpha^2D - \alpha^3D^2]B_{j-2}$
97	$\frac{1}{16}[-3j\alpha + 4j^2\alpha + 4j\alpha^2D + \alpha^3D^2]B_{j-3}$
98	$\frac{3}{8}[\alpha^2]C_{j-2}$
99	$\frac{1}{8}[-16\alpha + 17j\alpha - 4j^2\alpha + 8\alpha^2D - 4j\alpha^2D - \alpha^3D^2]B_{j-1}$
100	$\frac{1}{4}[-10j\alpha + 4j^2\alpha - 4\alpha^2D + 4j\alpha^2D + \alpha^3D^2]B_{j-2}$
101	$\frac{1}{8}[3j\alpha - 4j^2\alpha - 4j\alpha^2D - \alpha^3D^2]B_{j-3}$
102	$\frac{3}{2}[-\alpha^2]C_{j-2}$
103	$\frac{9}{4}[\alpha^2]C_{j-2}$

$$B_j = b^{(j)}_{\frac{3}{2}}(\alpha) \tag{99}$$

$$C_j = b^{(j)}_{\frac{5}{2}}(\alpha). \tag{100}$$

Note that these definitions differ from those of Brouwer and Clemence (1961).

ACKNOWLEDGMENTS

We thank Dima Fon-Der-Flaass for providing the proof of the identity given in Eq. (53) and Andrew Sinclair for pointing out the errors in the early printings of Brouwer and Clemence's expansion. Keren Ellis thanks the former Science and Engineering Research Council (SERC) for the award of a Postgraduate Studentship. Carl Murray is grateful to the SERC and the Particle Physics and Astronomy Research Council for the award of an Advanced Fellowship.

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