10 Second stars

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A summary of stellar structure equations (i.e. the equation of continuity, hydrostatic equilibrium, thermal equilibrium, energy transfer by radiation, by convection, nuclear reactions, convective mixing, the state equation, and the Saha equation):

$$\begin{aligned}
\frac{\mathrm{d}M_R}{\mathrm{d}R} &= 4\pi R^2 \rho\,, \\
\frac{\mathrm{d}P}{\mathrm{d}R} &= -\frac{GM_R\rho}{R^2}\,, \\
\frac{\mathrm{d}L_R}{\mathrm{d}R} &= 4\pi R^2 \rho \left(\epsilon_{\mathrm{nuc}} - \epsilon_\nu - T\frac{\mathrm{d}S}{\mathrm{d}t}\right)\,, \\
\left.\frac{\mathrm{d}T}{\mathrm{d}R}\right|_{\mathrm{rad}} &= -\frac{3\kappa\rho L_R}{16\pi acR^2T^3}\,, \\
\left.\frac{\mathrm{d}T}{\mathrm{d}R}\right|_{\mathrm{ad}} &= \frac{T}{P}\left(1-\frac{1}{\Gamma}\right)\frac{\mathrm{d}P}{\mathrm{d}R}\,. \\
\frac{\mathrm{d}T}{\mathrm{d}R} &= \max\left(\left.\frac{\mathrm{d}T}{\mathrm{d}R}\right|_{\mathrm{rad}}, \left.\frac{\mathrm{d}T}{\mathrm{d}R}\right|_{\mathrm{ad}}\right)\,, \\
\frac{\mathrm{d}(X,Y,Z)}{\mathrm{d}t} &= \sum_{(i,j,k)} \frac{\varepsilon_{(i,j,k)}}{\alpha_{(i,j,k)}}\,, \\
(X,Y,Z)_{\mathrm{C}} &= \frac{\int_{\mathrm{C}}(X,Y,Z)\mathrm{d}M_R}{\int_{\mathrm{C}}\mathrm{d}M_R}\,, \\
P &= \frac{\rho}{\mu m_{\mathrm{u}}}kT\lambda_{\mathrm{deg}}(\rho,T) + \frac{1}{3}aT^4\,, \\
\frac{x_j^{r+1}}{x_j^r}P_{\mathrm{e}} &= \frac{2(2\pi m_{\mathrm{e}})^{\frac{3}{2}}(kT)^{\frac{5}{2}}}{h^3}\frac{Z_j^{r+1}}{Z_j^r}\,\mathrm{e}^{-\frac{\chi_j^r}{kT}} \quad \text{for } \forall j,\,\forall r\,,
\end{aligned}$$

where M_R denotes the mass contained within a spherical volume with the radius R, ρ density, P pressure, L_R the radiative flux across a spherical surface with the radius R, $\epsilon_{\mathrm{nuc}}(\rho,T,X,Y,Z)$ specific power of nuclear reactions, ϵ_ν specific power of neutrinos, S specific entropy, T temperature, $\kappa(\rho,T,X,Y,Z)$ opacity, a radiative constant, c the speed of light, Γ adiabatic index, ε_i specific power of given reaction, α_i specific total energy; X, Y, Z abundances of hydrogen, helium and metals, μ mean molecular weight, m_{u} atomic mass constant, λ_{deg} degeneracy correction, x_j^r fraction of atoms j in ionisation state r, P_{e} electron pressure, m_{e} electron mass, h the Planck constant, Z_j partition function, and χ_j ionisation energy. Usually, the equations are rewritten to contain derivatives with respect to M_{R}, not R, but the form outlined above is easier to remember and interpret.

Summary: the evolutionary stellar-structure equations are a set of 7 integro–differential non-linear equations of the 2^{nd} order with mixed boundary conditions for 7 unknown functions $R(M_R,t)$, $P(M_R,t)$, $L_R(M_R,t)$, $T(M_R,t)$, $X(M_R,t)$, $Y(M_R,t)$, $Z(M_R,t)$ of 2 independent variables M_R, t.

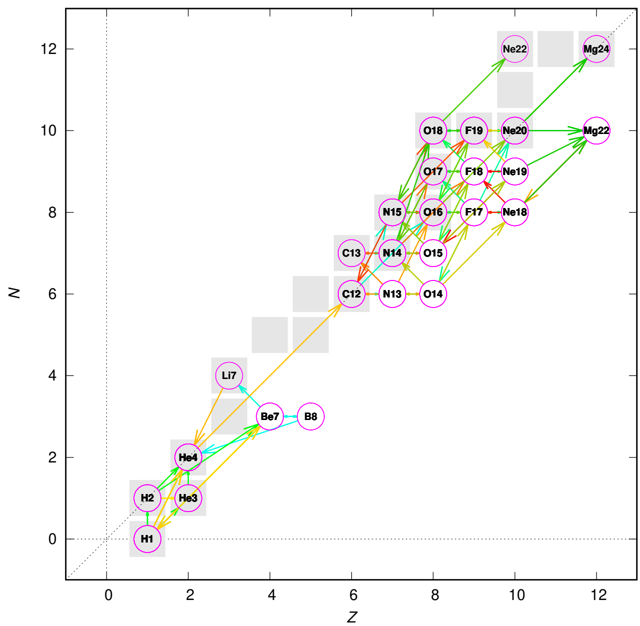

Fig. 1 — Thermonuclear reactions taken into account for stars with low and medium masses, shown on the proton number Z versus the neutron number N plot. The proton–proton chain, its 3 variants respectively, is on the bottom left, the CNO cycle on the upper right, the long arrow corresponds to the Salpeter reaction (also 3α). In total, there are 75 reactions. Stable isotopes are gray.

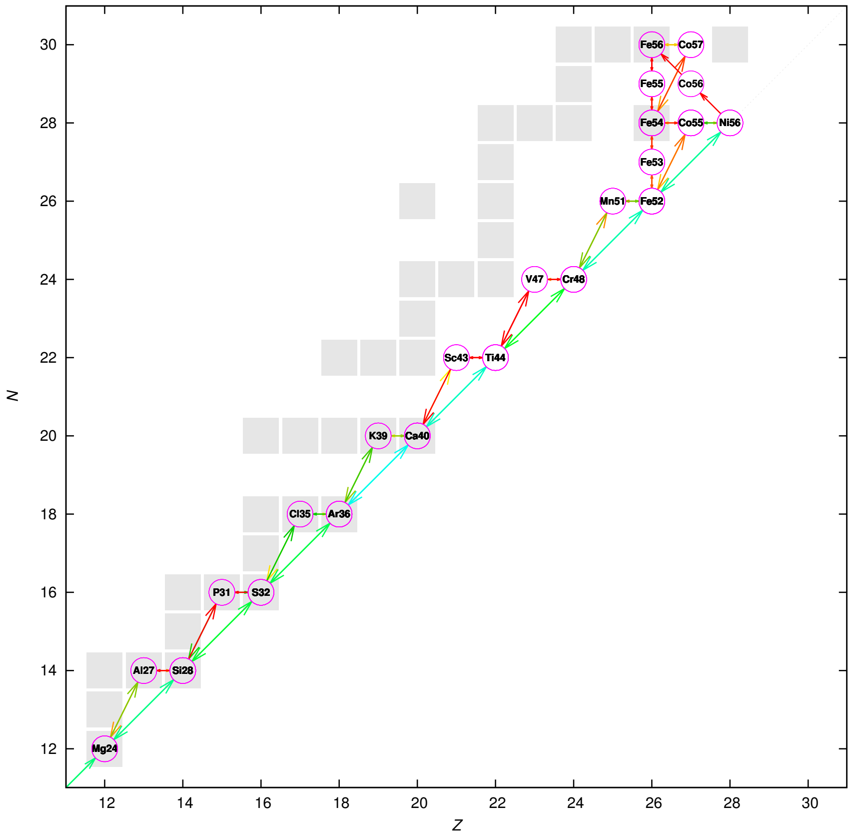

Fig. 2 — Reactions leading to the iron group (Fe, Co, Ni), which are taken into account for massive stars; the pp-chain and the CNO cycle (not shown) are often simplified. There are 91 reactions in total. Stable isotopes are gray. A possible nucleosynthesis by the s-process (capturing neutrons slower than β^- decay) is also not shown.

10.10 Numerical solution with the FVM method

One way of a numerical solution is an integration over (non-infinitesimal) volumes (FVM), in our case spherical shells delimited by selected values of M_j. Instead of unknown continuous functions, we have unknown discrete variables R_j, P_j, L_j, T_j, X_j, Y_j, Z_j for $j = 1..M$, whereas these are defined either inside the volume (P_j, T_j, X_j, Y_j, Z_j), or on its boundary (R_j, L_j). If we perform the simplest Euler discretisation in time at the same time:

$$\frac{\partial Y}{\partial t} \doteq \frac{Y^{n+1} - Y^n}{\delta t}, \tag{1}$$

etc. for other variables, and substitute the values at the new time t^{n+1} everywhere else, we obtain an implicit method, which requires a solution of a non-linear set of

$7M$ equations. Symbolically, we write all quantities at once as:

$$F_j = 0 \quad j = 1..7M\,. \tag{2}$$

The set cannot be inverted directly (it is still non-linear). If we substitute an estimate in it (e.g. a solution from the previous time step, a simple polytropic model) it surely will *not* be fulfilled:

$$F_j \neq 0\,.$$

However, we can approach the real solution by the Newton–Raphson method, which works for a one-dimensional function $f(x)$ as shown in Fig. 3.

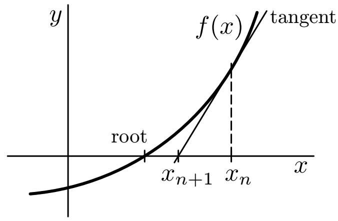

Fig. 3 — Scheme of one iteration of the Newton–Raphson method for searching roots of the function $f(x) = 0$. In point x_n, we construct a tangent with help of the derivative $f'(x_n)$, which intersection with axis x gives us new value x_{n+1}, which is closer to the root.

In our multi-dimensional case we actually search for many roots, one NR iteration represents finding of the intervals ΔX_k (i.e. corrections of estimates):

$$F_j + \sum_{k=1}^{7M} \overbrace{\frac{\partial F_j}{\partial X_k}}^{J_{jk}} \Delta X_k = 0 \qquad \text{pro } \forall j\,, \tag{3}$$

which consists in a solution of the linear set of equations; the matrix J_{jk} is called the Jacobian. It is very desirable the equations are neatly sorted, so that J_{jk} is a band matrix and its inversion simpler. Practically, we can perform the computation with the Mesastar program (Paxton aj. 2011, 2013, 2015).

If we would like to familiarize ourselves with where the equations are written in this program, we should start in file `star/job/run_star.f90` file and continue as follows:

```
do_run_star (1)
  run1_star (2)
    EVOLVE_LOOP:
      STEP_LOOP:
        star_evolve_step (3)
          do_evolve_step_part2 (4)
            IMPLICIT_MDOT_LOOP:
              do_struct_burn_mix (5)
                do_hydro_converge (6)
```

```
do_hydro_newton (7)
  hydro_newton_step (8)
    newt (9)
      newton (10)
        do_newton (11)
          eval_equations (12)
            eval_equ (13)
              eval_equ_for_solver (14)
                ZONES_LOOP:
                  do1_dlnd_dt_eqn (15)
                    e00 (16)
```

where one term of one derivative of one equation (the continuity) to the Jacobian occurs at the 16^{th} level of nesting (!).

10.11 Designations of evolutionary stages

In order to save some space, we denote evolutionary stages with the following abbreviations:

PS	... non-equilibrium protostar
pre-MS	... star before the main sequence
MS	... main sequence
SGB	... subgiant branch
RGB	... red giant branch
HB	... horizontal branch
AGB	... asymptotic giant branch
TP-AGB	... thermal pulses
post-AGB	... star after AGB
NB	... shell nebula
WD	... white dwarf
BD	... brown dwarf
RD	... red dwarf
BSG	... blue supergiant
RSG	... red supergiant
WR	... Wolf–Rayet star
SN	... supernova
NS	... neutron star
BH	... black hole

We will also use the letters corresponding to the Harvard spectral classification: O, B, A, F, G, K, M, L, T. Eventually, a finer subdivision by numbers O2 ... O8, B0 ... B8, etc.

Further, less common abbreviations are: WN i.e. WR with N lines; WC WR with C; LBV luminous blue variable, Of O with emission lines.

10.12 Evolutionary processes

Processes repeat themselves. Therefore, it is pointless to explain them again and again. We shall discuss them only once, and specify phases in which they usually occur.

1. *contraction → heating*: Every normal gas does heat up after compression, if they it is not in contact with a reservoir which would keep its temperature constant. For an adiabatic process ($\mathrm{d}Q = 0$) and the state equation holding at the same time we have $P = K\rho^\gamma = \rho/(\mu m_\mathrm{u})kT$, or $\rho^{\gamma-1}T^{-1} = k/(K\mu m_\mathrm{u})$, which for $\gamma > 1$ means heating during contraction. In a star, it is often a contraction of its core. It occurs in the SGB and RGB phases.
2. *expansion → cooling*: Of course, a reverse process is possible. It is often an expansion of the envelope; both processes occur at the same time, because after the core contraction something has to happen. Why the envelope does not contract too? If it could, it would had contracted even earlier, but now more thermal energy is incoming! It occurs again in the SGB, RGB.
3. *H burning in the centre*: Thermonuclear reactions as a positive add-on to $\mathrm{d}Q$ actually do not heat the star, but compensate for a loss (negative $\mathrm{d}Q$), and thus slooow down contraction. They keep the star very close to a hydrostatic equilibrium, all state quantities change very slowly. It is obviously the longest phase MS.
4. *increase of mean molecular weight μ*: A substantial change is a change of chemical composition, whilst in the state equation $P = \rho/(\mu m_\mathrm{u})kT$, there is μ in the denominator. For a pure fully ionised hydrogen, it would be $\mu \doteq 1/2$ (1 proton, 1 electron with $m \doteq 0$; 2 particles), for pure helium, $\mu \doteq 4/3$. This corresponds to a lower pressure P and also lower ∇P (although not necessarily). Matter changes "in our hands", it becomes softer. It is the major evolutionary process on the MS.
5. *decrease of opacity κ chemically*: Scattering on free electrons is by far the most important source of opacity in fully ionised plasma, where neither bound–free nor bound–bound transitions are possible. Consequently, κ decreases because electrons annihilate with positrons released by the pp chain or the CNO cycle. This exactly corresponds to the approximate Kramers opacity $\kappa \doteq 0{,}19\,\mathrm{cm}^2\,\mathrm{g}^{-1}$ $(1 - X)$, where the hydrogen abundance appears. It occurs also on the MS.
6. *H burning in the shell*: After decrease of $X_\mathrm{c} \to 0$, the maximum of the specific power ϵ_nuc moves outwards. This is called a shell burning. It continues almost uninterrupted from the SGB → AGB.
7. *increase of opacity κ by recombination*: Especially after the above-mentioned expansion/cooling, one expects bound–free and bound–bound transitions to increase κ. Every opaque fluid tends to convection. A convective zone then emerges (or extends), rises from the surface towards interior, until most of the volume is convective. It also causes a dredge-up of elements to the atmosphere. This happens during the RGB phase.

8. *emergence of degenerate core*: Electron gas at high densities (or equivalently at low temperatures) exhibits a degeneracy, in other words a steep increase of pressure due to the Pauli exclusion principle. ∇P would stop further collapse for low-mass stars; but there is not enough time for them to evolve. This situation nevertheless occurs in the RGB phase.
9. *cooling by neutrinos*: In certain brief phases, when the central temperature $T_{\rm c}$ is high, it can be $|\epsilon_\nu| \gg \epsilon_{\rm nuc}$. Inevitably, fast cooling means a fast collapse of the core. It happens between RGB/HB and in short phases TP-AGB, WR.
10. *He flash in the core*: Highly degenerate gas has a very special property: if we supply $\mathrm{d}Q$ T is not increased, P neither, there is no expansion, only a decrease of the degeneracy level ψ. Consequently, the He burning starts suddenly, until $\psi \to 0$. The respective Salpeter reaction (3α) starts off-center due to neutrino cooling. Although $L_{\rm nuc} \simeq 10^{10}\,\mathrm{W}$, it does not show-up too much at the surface. Again, it is a transition RGB/HB.
11. *He burning in the centre*: Surprisingly, the second source does *not* mean an increase of the luminosity L. One has to take into account an interaction of the sources — an expansion of the core causes an expansion of the H shell, a decrease of $\epsilon_{\rm H}$ and of the total L. Moreover, we expect a convective zone in the centre, in accord with the steep $\epsilon_{\rm nuc}(T)$. It is an equivalent of MS, but we distinguish it as HB
12. *He burning in the shell*: For $Y_{\rm c} \to 0$, the star evolves analogically, similarly as on the RGB, but this time we call the phase AGB. Both branches are located next to each other and they partially overlap. A second dredge-up also occurs in the atmosphere.
13. *He flashes in the shell*: Sequentially, the H and He shells become thinner, closer to each other, they interact, and due to degeneracy at the bottom of the He shell, there is a run-away burning of He and strong heating. In the upper part, a smallish convective zone occurs and the H shell temporarily shuts-off. Thermal pulses take approximately $10^2\,\mathrm{yr}$, intervals between the pulses $10^4\,\mathrm{yr}$. Eventually, a third dredge-up occurs if the convective zone connects with the outer one. This phase is denoted TP-AGB.
14. *mass loss*: Regardless whether it is a hydrodynamic wind, a radiation-driven wind, a superwind, an envelope pulsations, or some semi-empirical prescription, the star loses substantial mass in late evolutionary phases. In particular, during RGB, AGB, post-AGB, WR phases.
15. *C burning*: For massive stars, carbon reactions give rise to Ne, Mg isotopes in the core. Again, it occurs in the AGB phase, not to say a WD with O/Ne/Mg composition.
16. *emergence of the Fe core*: Further captures of α particles lead to formation of iron-group nuclei (Fig. 2), which have the maximum binding energy per nucleon. Inevitably, a core collapse occurs, with photodisintegration of Fe nuclei (back to α), neutronisation, an emergence of NS or BH, depending on a detailed structure of the core prior to the collapse.

17. *neutrino heating*: The matter density in the surroundings of the proto-NS is so high that energy is transported by neutrinos, which are thermalised during weak interactions with matter. Nucleosynthesis proceeds by the r-process (capturing neutrons faster than β^- decay) or p-process. One can expect some instabilities in doing so (see Sec. ??).

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10.13 Stellar evolution for various masses

For the solar-type stars with $M \geq 1{,}0\,M_\odot$ we expect a complex evolution (Fig. 4), complex changes of convective zones (Fig. 5), a He burning phase (Fig. 6) including flashes, which corresponds to the final stage of carbon/oxygen white dwarf:

MS G5 $\to$ SGB $\to$ RGB $\to$ HB $\to$ AGB $\to$ TP-AGB $\to$ post-AGB $\to$ NB $\to$ C/O WD

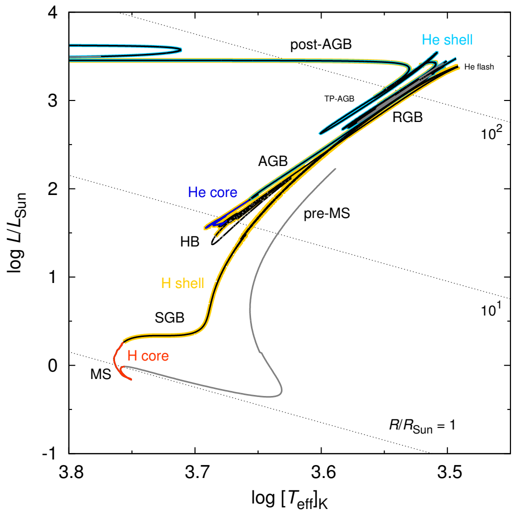

Fig. 4 — HR diagram for the Sun with the initial mass $M = 1\,M_\odot$. Black labels along the evolutionary track are individual phases, from pre-MS to post-AGB. Colours denote periods of H burning in the centre (red), H burning in the shell (yellow), He burning in the centre (blue), He burning in the shell (cyan). Constant-radius lines are plotted as dashed, according to $L = 4\pi R^2 \sigma T_{\rm eff}^4$. During the ascent along the RGB, there is a brief stopover, or the RGB dip. It is caused by the burning zone reaching a discontinuity in chemical composition, where the convective zone reached and induced an increase of X to $X_{\rm surf}$ $\to$ decrease of μ $\to$ decrease of T $\to$ temporary decrease of $\epsilon_{\rm nuc}$. For stars with lower metallicity we would expect an extensive blue loop (towards higher $T_{\rm eff}$) during the HB. The initial helium abundance was $Y = 0{,}274$, the metallicity $Z = 0{,}01954$, the opacities according to Grevesse and Noels (1993), the convective parameter $\alpha = 2{,}1$, parametric Reimers wind (in the RGB phase) with the efficiency $\eta = 0{,}6$, Blocker wind (AGB) with $\eta = 0{,}1$.

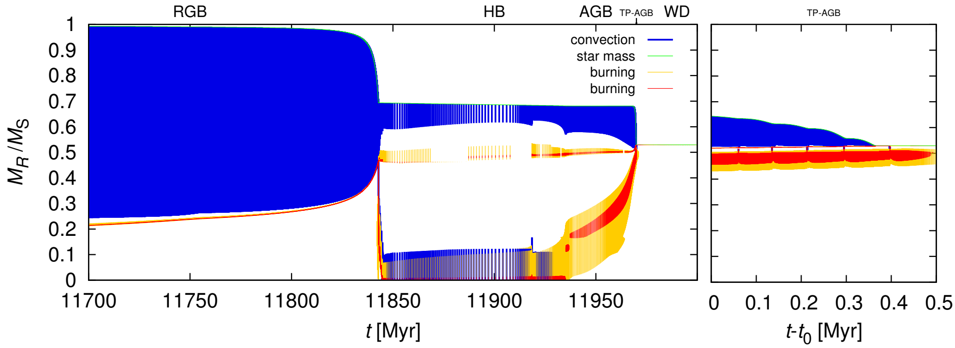

Fig. 5 — Kippenhahn diagram (for $M = 1\,M_\odot$) in late evolutionary phases (RGB to WD). There are at most 2 largest convective zones and 2 burning zones (H, He) denoted. The mass loss at the end of the RGB phase decreased the mass to 0,7 $M_\odot$, and later down to 0,5 $M_\odot$. The outer convective zone recedes in front of a thin H shell. The He flash occurs here at the time approximately 11,85 Gyr. At the end, both shells burn out as they move towards the surface. The detail on the right shows the TP-AGB phase, during which five He flashes occur in the shell, or thermal pulses. Temporary shut-offs of the H shell and smallish convective zones are at the edge of resolution.

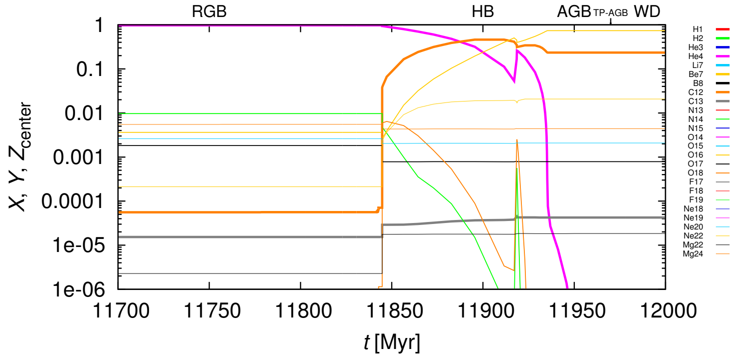

Fig. 6 — Abundances in the centre (for $M = 1\,M_\odot$) for all isotopes in the selected reaction net, including the pp chain, the CNO cycle and reactions up to ^{24}Mg. During the HB phase, He is consumed in the centre. The sudden increase of Y at the time 11,92 Gyr is related with the onset of O transformation to C and an extension of the central convective zone. In the last phase (WD) the core is composed mostly of oxygen ^{16}O and carbon ^{12}C.