Fyzika V

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Doporučená literatura:

W.S.C. Williams: Nuclear and Particle Physics

Přednáška 1 (2.10.2007)

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1.1. Jednotky

$$E^2 = (mc^2)^2 + (pc)^2$$

E eV, MeV, GeV

$$mc^2 \longrightarrow m$$

$$pc \longrightarrow p$$

$$E^2 = m^2 + p^2$$

$$\boxed{\begin{array}{l} neutron \\ m_n = 939{,}57\ MeV \\ proton \\ m_p = 938{,}27\ MeV \\ electron \\ m_e = 0{,}511\ MeV \\ \\ m_n - m_p = 1{,}3\ MeV \end{array}}$$

Coulombův zákon

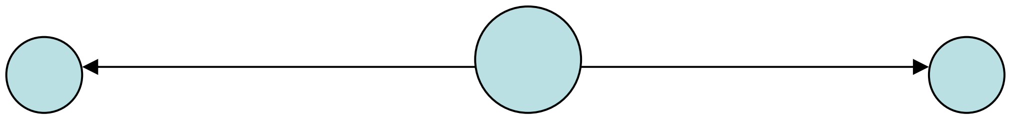
$$V(r) = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{r} = \frac{e^2}{4\pi\varepsilon_0\hbar c}\frac{\hbar c}{r} = \alpha\frac{\hbar c}{r}$$

α - konstanta jemné struktury

$$\alpha = 1/137{,}036 \cong 1/137$$

$$\hbar c = 197{,}327\ MeV \cdot fm$$

1.2. Kinematika



$$p_1 = p_2 = p^*$$

$$\sqrt{\left(p^*\right)^2 + m_1^2} + \sqrt{\left(p^*\right)^2 + m_2^2} = M$$

$$p^* = \frac{\sqrt{M^2 - (m_1 + m_2)^2}\sqrt{M^2 - (m_1 - m_2)^2}}{2M}$$

$$m_1 = m_2 = 0\,;\;\; p^* = M/2$$

$$m_1 = m_2 = m\,;\, p^* = \frac{\sqrt{M^2 - 4m^2}}{2} = \sqrt{M^2/4 - m^2}$$

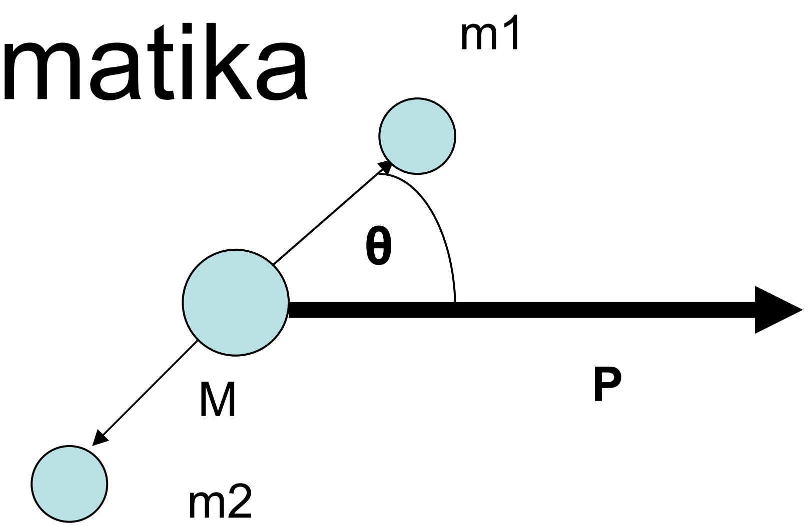
$$m_1 = m \;\; m_2 = 0\,;\, p^* = \frac{M^2 - m^2}{2M}$$

$$\pi^0 \to \gamma\gamma \;\; ;\;\; m_1 = m_2 = 0\,;\;\; p^* = M_{\pi^0}/2 = 135MeV/2 = 67{,}5MeV$$

$$\rho^0 \to \pi^+\pi^- \;\; ;\;\; m_1 = m_2 = m_\pi\,;\;\; p^* = \sqrt{M_\rho{}^2/4 - m_\pi{}^2} = \sqrt{770^2/4 - 140^2} \cong 359MeV$$

$$\pi^+ \to \mu^+\nu_\mu \;\; ;\;\; m_1 = m_\mu \; m_2 = 0\,;\;\; p^* = \frac{M_\pi{}^2 - m_\mu{}^2}{2M_\pi} = \frac{140^2 - 106^2}{2 \cdot 140} \cong 30MeV$$

1.2. Kinematika



$$e_1^* = \sqrt{\left(p^*\right)^2 + m_1^2}$$

$$p^*{}_{1,\parallel} = p^* \cos(\theta)$$

$$p^*{}_{1,\perp} = p^* \sin(\theta)$$

$$e_1^{lab} = \gamma(e_1^* + \beta p^* \cos(\theta)) \qquad \gamma = \frac{\mathrm{E}}{\mathrm{M}} = \frac{\sqrt{P^2 + M^2}}{M} \quad \beta = \frac{\mathrm{P}}{\mathrm{E}} = \frac{P}{\sqrt{P^2 + M^2}}$$

$$p^{lab}{}_{1,\parallel} = \gamma(p^* \cos(\theta) + \beta e_1^*)$$

$$p^{lab}{}_{1,\perp} = p^*{}_{1,\perp} = p^* \sin(\theta)$$

1.3. Atom vodíku - opakování

$$\left(-\frac{(\hbar c)^2}{2m_e c^2}\Delta_e-\frac{(\hbar c)^2}{2m_p c^2}\Delta_p-\alpha\frac{\hbar c}{\left|\vec{r}_e-\vec{r}_p\right|}\right)\varphi\left(\vec{r}_e,\vec{r}_p\right)=E_{tot}\varphi\left(\vec{r}_e,\vec{r}_p\right)$$

$$\vec{R}=\frac{m_e\vec{r}_e+m_p\vec{r}_p}{m_e+m_p}\qquad \vec{r}=\vec{r}_e-\vec{r}_p\qquad m_{e,p}c^2\longrightarrow m_{e,p}$$

$$\left(-\frac{(\hbar c)^2}{2(m_e+m_p)}\Delta_R-\frac{(\hbar c)^2}{2m_{ep}}\Delta_r-\alpha\frac{\hbar c}{\left|\vec{r}\right|}\right)\varphi\left(\vec{R},\vec{r}\right)=E_{tot}\varphi\left(\vec{R},\vec{r}\right)$$

$$m_{ep}=\frac{m_e m_p}{m_e+m_p}\cong m_e\;;\;\varphi\left(\vec{R},\vec{r}\right)=\Psi\left(\vec{R}\right)\psi\left(\vec{r}\right);\,E_{tot}=E_R+E$$

$$-\frac{(\hbar c)^2}{2(m_e+m_p)}\Delta_R\Psi\left(\vec{R}\right)=E_R\Psi\left(\vec{R}\right);\;\boxed{\left(-\frac{(\hbar c)^2}{2m_{ep}}\Delta_r-\alpha\frac{\hbar c}{\left|\vec{r}\right|}\right)\psi\left(\vec{r}\right)=E\psi\left(\vec{r}\right)}$$

1.3. Atom vodíku - opakování

$$\left(-\frac{(\hbar c)^2}{2m_{ep}}\Delta_r - \alpha\frac{\hbar c}{\left|\vec{r}\right|}\right)\psi(\vec{r}) = E\psi(\vec{r})$$

Laplacian: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right) + \frac{1}{r^2\sin\vartheta^2}\frac{\partial^2}{\partial\phi^2}$$

$$\psi(\vec{r}) = R(r)Y_{lm}(\theta,\varphi)$$

$$\hat{L}^2\,Y_{lm}(\theta,\varphi) = \hbar^2 l(l+1)Y_{lm}(\theta,\varphi);\ \hat{L}_z\,Y_{lm}(\theta,\varphi) = \hbar m Y_{lm}(\theta,\varphi)$$

$$-\frac{(\hbar c)^2}{2m_e}\frac{d^2}{dr^2}(rR(r)) - \alpha\frac{\hbar c}{r}(rR(r)) + \frac{(\hbar c)^2 l(l+1)}{2m_e r^2}(rR(r)) = E(rR(r))$$

$$u(r) = rR(r)\xrightarrow[r\to 0]{} r^{l+1} \quad ; \quad u(r) = rR(r)\xrightarrow[r\to\infty]{} 0$$

1.3. Atom vodíku - opakování

$$l = 0$$

$$-\frac{(\hbar c)^2}{2m_e}\frac{d^2}{dr^2}u(r) - \alpha\frac{\hbar c}{r}u(r) = Eu(r)$$

$$u(r) = re^{-r/r_B}\ ;\ \frac{d^2}{dr^2}u(r) = \frac{1}{r_B^2}u(r) - \frac{2}{rr_B}u(r)$$

$$-\frac{(\hbar c)^2}{2m_e}\frac{1}{r_B^2}u(r) + \frac{(\hbar c)^2}{2m_e}\frac{2}{rr_B}u(r) - \alpha\frac{\hbar c}{r}u(r) = Eu(r)$$

$$\frac{(\hbar c)^2}{2m_e}\frac{2}{rr_B}u(r) - \alpha\frac{\hbar c}{r}u(r) = 0 \quad ; \quad -\frac{(\hbar c)^2}{2m_e}\frac{1}{r_B^2}u(r) = Eu(r)$$

$$r_B = \frac{(\hbar c)^2}{2m_e}\frac{2}{\alpha\hbar c} = \frac{\hbar c}{m_e\alpha} = \frac{197 MeVfm}{0{,}511 MeV/137} = 52816\,fm$$

$$E = -\frac{(\hbar c)^2}{2m_e}\frac{1}{r_B^2} = -\frac{(\hbar c)^2}{2m_e}\frac{(m_e\alpha)^2}{(\hbar c)^2} = -\frac{1}{2}m_e\alpha^2 = -\frac{1}{2}0{,}511 MeV\frac{1}{137^2} = -13{,}6 eV$$

1.3. Atom vodíku - opakování

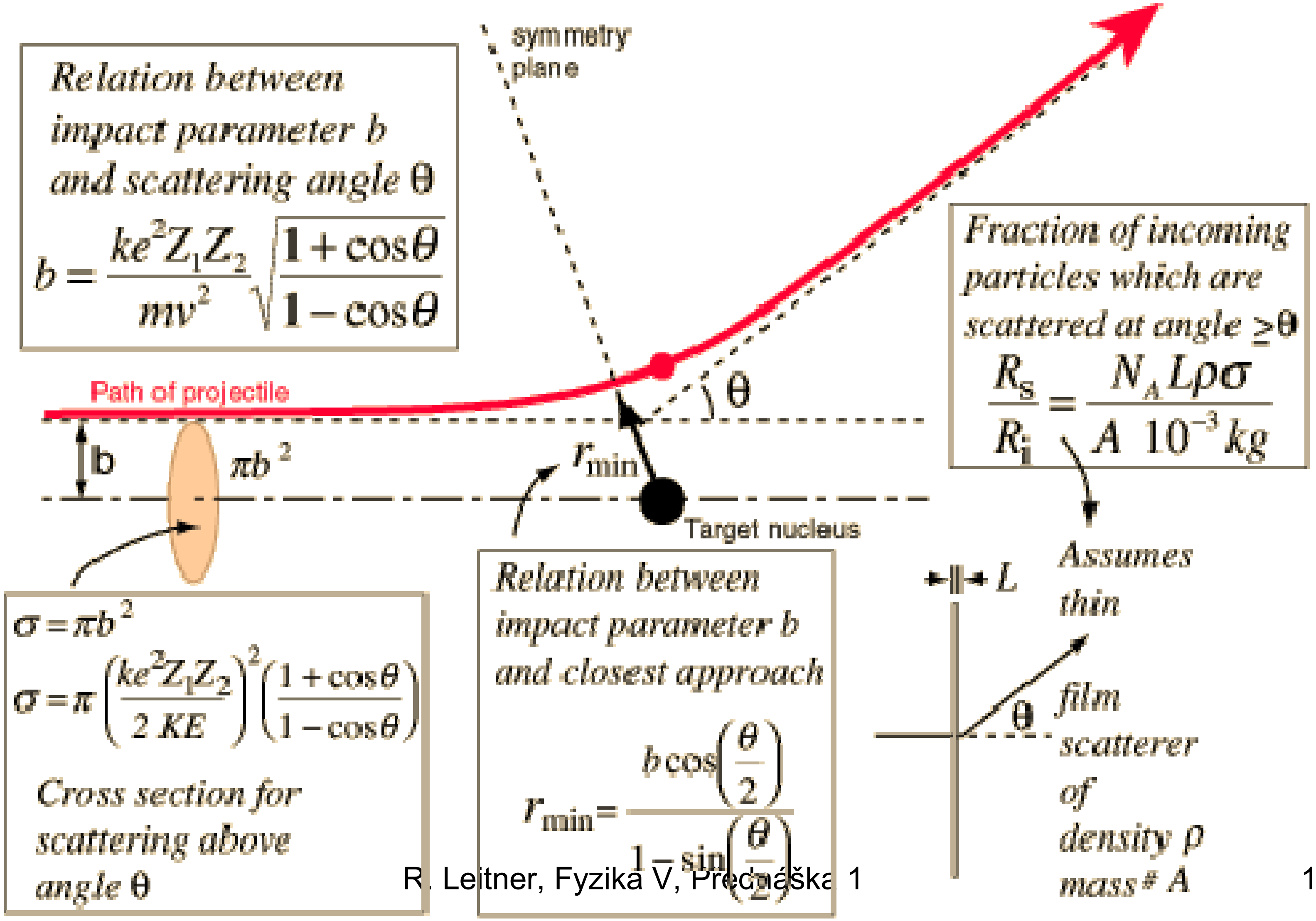
$$E = -\frac{1}{2} m_e \alpha^2 \; ; E = T + V = \frac{1}{2} m_e \alpha^2 - 2\frac{1}{2} m_e \alpha^2 = 13{,}6 eV - 27{,}2 eV$$

$$T = \frac{1}{2} m_e \left(\frac{v}{c}\right)^2 = \frac{1}{2} m_e \beta^2 \quad ; \quad \text{rychlost elektronu } \beta = \frac{1}{137} = \alpha$$

atom s Z protony :

$$E = -\frac{1}{2} m_e (Z\alpha)^2 = Z^2 13{,}6 eV \; ; E_{Fe} = -26^2 13{,}6 eV = -9{,}19 keV$$

1.4. Rutherfordův rozptyl



1.4. Rutherfordův rozptyl klasicky - opakování

$$b(\theta)=\frac{b(\pi/2)}{\tan(\theta/2)}\,;\ \mathrm{b}(0)\to\infty\,;\ \mathrm{b}(\pi)\to 0$$

$$b(\pi/2)=r_{\min}/2\,;\ \mathrm{T_k}=\alpha Z_1 Z_2\frac{\hbar c}{r_{\min}};\ r_{\min}=\alpha Z_1 Z_2\frac{\hbar c}{T_k}$$

$$r_{\min}=\frac{1}{137}2\cdot 79\cdot\frac{197\,MeVfm}{7{,}7\,MeV}=29{,}5\,fm$$

$$\sigma(\theta\geq\theta_0)=\pi b^2(\theta)$$

$$\sigma(\theta\geq\pi/2)=\pi b^2(\pi/2)=\pi\frac{r^2{}_{\min}}{4}=3{,}14\frac{(29{,}5\,fm)^2}{4}=683\,fm^2=$$

$$683\cdot 10^{-26}cm^2=6{,}83\cdot 10^{-24}cm^2=6{,}83\,barn$$

1.4. Rutherfordův rozptyl klasicky - opakování

$$dP(x) = (1 - P(x))\sigma[cm^2]dx[cm]\rho[g/cm^3]N_A[1/mol]/A[g/mol]$$

$$-d(1 - P(x)) = (1 - P(x))\frac{\sigma\rho N_A}{A}dx$$

$$1 - P(L) = e^{-\frac{\sigma\rho N_A}{A}L}$$

$$\boxed{P(L) = 1 - e^{-\frac{\sigma\rho N_A}{A}L} \cong 1 - (1 - \frac{\sigma\rho N_A}{A}L) = \frac{\sigma\rho N_A}{A}L}$$

$$\sigma = 6{,}83b; \rho = 20g/cm^3; N_A = 6{,}023 \cdot 10^{23}; L = 500nm; A = 197$$

$$P(\theta > 90^0, 500nmAu) = \frac{6{,}83 \cdot 10^{-24}cm^2 20g/cm^3 6{,}023 \cdot 10^{23}mol^{-1}}{197g/mol}500 \cdot 10^{-7}cm =$$

$$2{,}09 \cdot 10^{-5} \cong \frac{1}{50000}$$

1.4. Rutherfordův rozptyl klasicky - opakování

Diferenciální účinný průřez

$$d\sigma = 2\pi b(\theta)db = 2\pi b(\theta)\left|\frac{db}{d\theta}\right|d\theta$$

$$b(\theta) = \frac{b(\pi/2)}{\tan(\theta/2)};\ \frac{db}{d\theta} = -\frac{b(\pi/2)}{\tan^2(\theta/2)}\frac{1}{\cos^2(\theta/2)}\frac{1}{2}$$

$$d\sigma = -2\pi\frac{b(\pi/2)}{\tan(\theta/2)}\frac{b(\pi/2)}{\tan^2(\theta/2)}\frac{1}{\cos^2(\theta/2)}\frac{1}{2}d\theta$$

$$\frac{d\sigma}{d\theta} = -\frac{\pi(r_{\min}/2)^2}{\sin^3(\theta/2)}\cos(\theta/2) = -\frac{1}{2}\frac{\pi(r_{\min}/2)^2}{\sin^4(\theta/2)}2\sin(\theta/2)\cos(\theta/2) = -\frac{1}{2}\frac{\pi(r_{\min}/2)^2}{\sin^4(\theta/2)}\sin(\theta)$$

$$\frac{d\sigma}{-\sin(\theta)d\theta} = \boxed{\frac{d\sigma}{d\cos(\theta)} = \frac{\pi(r^2{}_{\min}/8)}{\sin^4(\theta/2)}}$$

1.4. Výsledek Rutherfordova pokusu

- Částice alpha se rozptylují jakoby jádro bylo bodové
- Rozměr jádra je tudíž menší než Rmin=30 fm, tj. asi 2000 krát menší než rozměr atomů
- Alfa částice pronikaly do lehkých jader (Be) a tudíž jádro má nejspíš konečný rozměr.

1.5. Definice účinného průřezu v kvantové mechanice

$$d\sigma = \frac{\overrightarrow{j_{out}} \cdot d\vec{S}}{j_{in}} \;;\; \vec{j} = \frac{\hbar}{i2m}\left(\psi^* \vec{\nabla}\psi - \psi \vec{\nabla}\psi^*\right);$$

$$\psi_{\text{in}} = e^{i\frac{pz}{\hbar c}} \;;\; j_{in,z} = \frac{\hbar}{i2m}\left(e^{-i\frac{pz}{\hbar c}} i\frac{p}{\hbar c} e^{i\frac{pz}{\hbar c}} - e^{i\frac{pz}{\hbar c}}\left(-i\frac{p}{\hbar c}\right)e^{-i\frac{pz}{\hbar c}}\right) = \frac{\hbar}{i2m} 2i\frac{p}{\hbar c} = \frac{p}{mc^2} = \frac{p}{m}$$

$$\psi_{\text{out}} = f(\theta)\frac{e^{i\frac{pr}{\hbar c}}}{r} ; j_{out,x} = \frac{\hbar}{i2m}\left(f^*(\theta)\frac{e^{-i\frac{pr}{\hbar c}}}{r} i\frac{p}{\hbar c} f(\theta)\frac{e^{i\frac{pr}{\hbar c}}}{r}\frac{\partial r}{\partial x} - f(\theta)\frac{e^{i\frac{pr}{\hbar c}}}{r}\left(-i\frac{p}{\hbar c}\right) f^*(\theta)\frac{e^{-i\frac{pr}{\hbar c}}}{r}\frac{\partial r}{\partial x}\right) =$$

$$\frac{\hbar}{i2m} f^*(\theta) f(\theta) 2i\frac{p}{\hbar c}\frac{x}{r} = \frac{p}{mc^2}\left|f(\theta)\right|^2 \frac{x}{r}$$

$$\overrightarrow{j_{out}} = \frac{p}{m}\frac{\left|f(\theta)\right|^2}{r^2}\frac{\vec{r}}{r}$$

$$d\sigma = \frac{\overrightarrow{j_{out}} \cdot d\vec{S}}{j_{in}} = \frac{\frac{p}{m}\frac{\left|f(\theta)\right|^2}{r^2}\frac{\vec{r}}{r} \cdot r^2 d\Omega \frac{\vec{r}}{r}}{\frac{p}{m}} = \left|f(\theta)\right|^2 d\Omega$$

$$\boxed{\frac{d\sigma}{d\cos(\theta)d\phi} = \left|f(\theta)\right|^2}$$

1.5. Rozptyl v kvantové mechanice

$$\left(\Delta+\left(\frac{\sqrt{2ME}}{\hbar c}\right)^2\right)\psi^{(0)}(\vec{r})=0 \;\; ; \;\; \psi^{(0)}(\vec{r})=e^{+ikz} \;\; ; \;\; k=\frac{\sqrt{2ME}}{\hbar c}$$

$$\left(-\frac{(\hbar c)^2}{2M}\Delta+V(\vec{r})\right)\psi(\vec{r})=E\cdot\psi(\vec{r}) \;\; ; \;\; \left(\Delta+\left(\frac{\sqrt{2ME}}{\hbar c}\right)^2\right)\psi(\vec{r})=\frac{2M}{(\hbar c)^2}V(\vec{r})\cdot\psi(\vec{r}) \;\; ; \;\; \left(\Delta+k^2\right)\psi(\vec{r})=\frac{2M}{(\hbar c)^2}V(\vec{r})\cdot\psi(\vec{r})$$

$$\left(\Delta+k^2\right)\psi^{(n)}(\vec{r})=\frac{2M}{(\hbar c)^2}V(\vec{r})\cdot\psi^{(n-1)}(\vec{r}) \;\; ; \;\; \psi^{(n)}(\vec{r})=-\frac{2M}{(\hbar c)^2}\int V(\vec{r}\,')\cdot\psi^{(n-1)}(\vec{r}\,')\cdot G(\vec{r}-\vec{r}\,')\cdot d^3\vec{r}\,'$$

$$\left(\Delta+k^2\right)G(\vec{r}-\vec{r}\,')=\delta(\vec{r}-\vec{r}\,') \quad ; \quad G(\vec{r}-\vec{r}\,')=\frac{1}{4\pi}\frac{e^{ik\left|\vec{r}-\vec{r}\,'\right|}}{\left|\vec{r}-\vec{r}\,'\right|}$$

$$\psi^{(1)}(\vec{r})=-\frac{M}{2\pi(\hbar c)^2}\int V(\vec{r}\,')\cdot\psi^{(0)}(\vec{r}\,')\cdot\frac{e^{ik\left|\vec{r}-\vec{r}\,'\right|}}{\left|\vec{r}-\vec{r}\,'\right|}d^3\vec{r}\,'+\psi^{(0)}(\vec{r})$$

$$\psi^{(2)}(\vec{r})=-\frac{M}{2\pi(\hbar c)^2}\int V(\vec{r}\,')\cdot\psi^{(1)}(\vec{r}\,')\cdot\frac{e^{ik\left|\vec{r}-\vec{r}\,'\right|}}{\left|\vec{r}-\vec{r}\,'\right|}d^3\vec{r}\,'=-\frac{M}{2\pi(\hbar c)^2}\int V(\vec{r}\,')\cdot\left(-\frac{M}{2\pi(\hbar c)^2}\int V(\vec{r}\,'')\cdot\psi^{(0)}(\vec{r}\,'')\cdot\frac{e^{ik\left|\vec{r}\,'-\vec{r}\,''\right|}}{\left|\vec{r}\,'-\vec{r}\,''\right|}d^3\vec{r}\,''+\psi^{(0)}(\vec{r}\,')\right)\cdot\frac{e^{ik\left|\vec{r}-\vec{r}\,'\right|}}{\left|\vec{r}-\vec{r}\,'\right|}d^3\vec{r}\,'+\psi^{(0)}(\vec{r})=$$

$$-\frac{M}{2\pi(\hbar c)^2}\int V(\vec{r}\,')\cdot\left(-\frac{M}{2\pi(\hbar c)^2}\int V(\vec{r}\,'')\cdot\psi^{(0)}(\vec{r}\,'')\cdot\frac{1}{4\pi}\cdot\frac{e^{ik\left|\vec{r}\,'-\vec{r}\,''\right|}}{\left|\vec{r}\,'-\vec{r}\,''\right|}d^3\vec{r}\,''\right)\cdot\frac{1}{4\pi}\cdot\frac{e^{ik\left|\vec{r}-\vec{r}\,'\right|}}{\left|\vec{r}-\vec{r}\,'\right|}d^3\vec{r}\,'+\psi^{(1)}(\vec{r})$$

1.5. Rozptyl v kvantové mechanice

$$\left|r - r'\right| = \sqrt{\left(\vec{r} - \vec{r}'\right)\left(\vec{r} - \vec{r}'\right)} \cong \sqrt{\vec{r}\vec{r} - 2\vec{r}\vec{r}'} = \sqrt{\vec{r}\vec{r}\left(1 - 2\vec{r}\vec{r}' / \vec{r}\vec{r}\right)}$$

$$\cong \sqrt{\vec{r}\vec{r}}\left(1 - \vec{r}\vec{r}' / \vec{r}\vec{r}\right) = r - \vec{r}'\vec{n}$$

$$\frac{e^{ik\left|\vec{r} - \vec{r}'\right|}}{\left|\vec{r} - \vec{r}'\right|} \cong \frac{e^{ikr}}{r} \cdot \frac{e^{-ik\vec{r}'\vec{n}}}{(1 - \vec{r}'\vec{n} / r)} \cong \frac{e^{ikr}}{r} \cdot e^{-ik\vec{r}'\vec{n}}$$

$$\boxed{\begin{aligned} \psi^{(1)}(r) &= \left(-\frac{M}{2\pi(\hbar c)^2} \int e^{-ik\vec{r}'\vec{n}} \cdot V(r') \cdot e^{ikz'} \cdot d^3 r' \right) \frac{e^{+ikr}}{r} = f(\theta) \frac{e^{+ikr}}{r} \\ f(\theta, \phi) &= f^{(1)}(\theta, \phi) = -\frac{M}{2\pi(\hbar c)^2} \int e^{-ik\vec{r}'\vec{n}} \cdot V(r') \cdot e^{ikz'} \cdot d^3 r' \end{aligned}}$$

1.5. Rozptyl v kvantové mechanice

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\cos(\theta)d\phi} = \left| -\frac{M}{2\pi(\hbar c)^2} \int e^{-i(\overrightarrow{k_f} - \overrightarrow{k_i})\vec{r}} \cdot V(r') \cdot d^3 r' \right|^2 =$$

$$\frac{M^2}{4\pi^2(\hbar c)^4} \left| \int e^{-i\frac{(\overrightarrow{p_f} - \overrightarrow{p_i})\vec{r}}{\hbar c}} \cdot V(r') \cdot d^3 r' \right|^2 = \frac{M^2}{4\pi^2(\hbar c)^4} \left| \int e^{-i\frac{\vec{q}\vec{r}}{\hbar c}} \cdot V(r') \cdot d^3 r' \right|^2$$

$$\left[\frac{d\sigma}{d\Omega}\right] = \frac{GeV^2}{GeV^4 fm^4} \left(GeV \cdot fm^3\right)^2 = fm^2$$

$$\left| f(\theta) \right|^2 = \frac{M^2}{4\pi^2(\hbar c)^4} \left| \int e^{-i\frac{\vec{q}\vec{r}}{\hbar c}} \cdot V(r') \cdot d^3 r' \right|^2$$