LAWSON

LIMITS NEWTON

GRAVITATIONAL RADIUS → SUN !

DENSITY → SUN !

TWO TYPES BLACK HOLE (HAWKING)

COMPACTNES – MEASUREMENTS

MINKOWSKI → GENERAL RELATIVITY

COVARIANT DERIVATIVE

COMMUTATOR, RIEMANN, RICCI,

VACUUM EINSTEIN FIELD EQUATIONS

SCHWARZSCHILD

SYMMETRIAS, KILLINGS

$(\eta\eta) = g_{tt}$

COMMUTATORS

MOTION

GEODESIC MOTION

CONSERVATION ENERGY, ANG. MOM.

SPECIFIC

COORDINATE $\ell = -u_\varphi / u_t$ ETC

EFFECTIVE POTENTIAL NEWTON

EPICICLIC NEWTON ← WHEELER'S PRINC.

EFFECTIVE, EPICYCLIC EINSTEIN

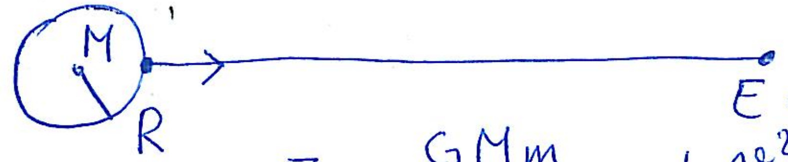
FOR SCHWARZSCHILD

ISCO : THE MOST IMPORTANT ITEM.

LAWSON * * * First Lecture

Thursday, October 1, Mathematics, Charles University

Limits of applicability of Newton's theory: $v < c$, v = escape velocity.



$E = 0,\ v = 0$

$$E = -\frac{GMm}{r} + \frac{1}{2} v^2 m$$

$$v^2 = \frac{2GM}{r} \quad ; \quad \frac{v^2}{c^2} = \underbrace{\left(\frac{2GM}{c^2}\right)}_{\equiv R_G} \frac{1}{R}$$

gravitational radius (Schwarzschild)

$$\boxed{\frac{R_G}{R} \ll 1}$$

compactness

Density of an object with compactness ≈ 1

$$\rho_* \approx \frac{M}{R_G^3} = \frac{Mc^6}{8G^3M^3}$$

$$\approx \left(\frac{c^6}{8G^3 M_\odot^2}\right)\left(\frac{M_\odot}{M}\right)^2$$

$$\boxed{\rho_* \approx 7 \times 10^{16} \left[\frac{g}{cm^3}\right] \left(\frac{M_\odot}{M}\right)^2}$$

VERTE !

$$\frac{c^6}{8G^3 M_\odot^2} = \frac{M_\odot}{(R_G^\odot)^3} = \frac{10^{33}\, g}{3^3 \cdot (10^5)^3\, cm^3} =$$

$$= \frac{10^{33}}{27 \cdot 10^{15}} = \frac{1}{2.7}\, 10^{17} \approx 4 \times 10^{6}$$

$$\begin{array}{r} 33 \\ 16 \\ \hline 17 \end{array}$$

$$R_G^\odot \approx 3\,km$$

$$M^\odot \approx 10^{33}\, g$$

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$$\rho_* = \begin{cases} 10^{15}\,\mathrm{g/cm^3} & \text{for } M \sim 10 M_\odot \\ 10^{0}\,\mathrm{g/cm^3} & \text{for } M \sim 10^8 M_\odot \end{cases}$$

Chandra, Zwicky, Landau, Oppenheimer

"stellar mass" compact objects (collapse)

"super-massive" compact objects (collection)

Laplace, Mitchel

THESE ARE

$\boxed{\text{Black holes}}$ have been "found" by astronomers

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The only reasoning: compactness $\boxed{\text{NOTE}}$

One needs to measure mass and size

Mass "easy" (Kepler)

Size tricky (e.g. variability, spectral fitting, direct images

The contribution of Prague group $\boxed{\text{NOTE}}$

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Einstein's General Relativity: Theory of Gravity

$c = 1 = G$ ← BE AWARE!

Special Relativity, Minkowski spacetime

$t+dt,\ x+dx,\ y+dy,\ z+dz$

ds

t, x, y, z

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

note minus

"signature" $+ - - -$

in <u>many</u> textbooks

$+ - - -$

Flat Space, Cartesian coordinates

$$dl^2 = dx^2 + dy^2 + dz^2$$

Spherical coordinates

$$dl^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

$$\left.\begin{aligned} x &= r\sin\theta\sin\varphi \\ y &= r\sin\theta\cos\varphi \\ z &= r\cos\theta \end{aligned}\right\}$$

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Curved space, curved spacetimes:

$$\boxed{ds^2 = g_{ik}\, dx^i dx^k}$$

Q scalars

Q^i vectors

Q^{ik} tensors

∇_i (covariant) derivative

$$\nabla_i Q \equiv \partial_i Q \equiv \frac{\partial Q}{\partial x^i}$$

$$\nabla_i Q^k = \partial_i Q^k + \Gamma^k_{ij} Q^j$$

$$\nabla_i Q_k = \partial_i Q_k - \Gamma^j_{ik} Q_j$$

$$\Gamma^j_{ik} = \frac{g^{js}}{2}\left(\frac{\partial g_{is}}{\partial x^k} + \frac{\partial g_{ks}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^s}\right)$$

Christoffel symbol

$$\boxed{\nabla_i g_{jk} = 0}$$ metric is covariantly constant!

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Commutator

$$\partial_i \partial_k \mathcal{Q} - \partial_k \partial_i \mathcal{Q} = 0$$

$$\nabla_i \nabla_k \mathcal{Q}^j - \nabla_k \nabla_i \mathcal{Q}^j = R^j_{\ isk} \mathcal{Q}^s \neq 0$$

Riemann tensor

Riemann tensor is built from g_{ik} and its derivatives, up to second derivatives

$$R_{ik} = R^j_{\ ijk}$$ Ricci tensor

$$\boxed{R_{ik} = 0}$$ ← Einstein's field equations for the vacuum case.

$R_{ik}=0$ Valid for all black holes

Spacetime fixed: black hole astrophysics

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Schwarzschild spacetime:

$$ds^2 = g_{ik}\,dx^i dx^k =$$
$$= \left(1 - \frac{R_G}{r}\right) dt^2 - \left(1 - \frac{R_G}{r}\right)^{-1} dr^2$$
$$- r^2 \left[d\theta^2 + \sin^2\theta\, d\varphi^2 \right]$$

Note: $\partial_t g_{ik} = 0 = \partial_\varphi g_{ik}$

stationary $\partial_t = 0$

axially symmetric $\partial_\varphi = 0$

All black holes have these symmetries.

Killing vectors

$\partial_t = 0 \qquad \nabla_i \eta_k + \nabla_k \eta_i = 0; \quad \eta^k = \delta^k_t$

$\partial_\varphi = 0 \qquad \nabla_i \xi_k + \nabla_k \xi_i = 0; \quad \xi^k = \delta^k_\varphi$

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Several useful relations

$$(\eta\eta) = (\eta^i \eta_i) = \eta^i \eta^k g_{ik} = \delta^i_t \delta^k_t g_{ik} = g_{tt}$$

$$(\eta\xi) = (\eta^i \xi_i) = g_{t\varphi} = 0 \quad \text{(in Schwarzschild)}$$

$$(\xi\xi) = (\xi^i \xi_i) = g_{\varphi\varphi}$$

$$\boxed{\eta^i \nabla_i \xi_k = \xi^i \nabla_i \eta_k} \leftarrow \text{valid in Schwarzschild}$$

$$\eta^i \nabla_i \xi_k = -\frac{1}{2} \nabla_k (\xi\eta)$$

Proof:

$$\nabla_k (\eta^i \xi_i) = \eta^i \nabla_k \xi_i + \xi_i \nabla_k \eta^i =$$

$$= \eta^i \nabla_k \xi_i + \xi^i \nabla_k \eta_i = -\eta^i \nabla_i \xi_k - \xi^i \nabla_i \eta_k$$

$$= -\eta^i \nabla_i \xi_k - \eta^i \nabla_i \xi_k = \underline{-2\eta^i \nabla_i \xi_k}$$

qed.

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Motion of particles

$x^i = x^i(s)$ spacetime trajectory

$u^i = \frac{dx^i}{ds}$ spacetime Four-velocity

$$(uu) = u^i u^k g_{ik} = \frac{dx^i}{ds}\frac{dx^k}{ds} g_{ik} = \frac{ds^2}{ds^2} = 1$$

$\boxed{\text{Four velocity } (uu) = 1 \text{ is a \underline{unit} timelike vector}}$

Free geodesic motion

$$a_k = u^i \nabla_i u_k = 0$$

acceleration

$a_k u^k = 0$ proof:

$$a_k u^k = u^k u^i \nabla_i u_k = u^i (u^k \nabla_i u_k) = 0$$

qed.

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Energy & angular momentum conservation

$\boxed{E \equiv u^i \eta_i}$ is conserved along the geodesic motion (Energy)

$$u^k \nabla_k (u^i \eta_i) = \underbrace{\underbrace{u^i u^k}_{\text{symmetric}} \underbrace{\nabla_k \eta_i}_{\text{antysymmetric}}}_{=0} + \eta_i \underbrace{u^k \nabla_k u^i}_{\text{geodesic} = 0}$$

$$\boxed{u^k \nabla_k E = 0}$$

geo.

For the same reason:

$$u^k \nabla_k L = 0 \qquad L \equiv - u^i \xi_i$$

and

$$l = \frac{L}{E} \quad \text{specific angular momentum}$$

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<u>Note:</u> $E = u^i \eta_i = \eta^i u_i \overset{*}{=} \delta^i_t u_i = u_t$

$-L = u^i \xi_i = u_\varphi$

$$\left.\begin{aligned} E &= u_t \\ L &= -u_\varphi \\ \frac{L}{E} &= -\frac{u_\varphi}{u_t} = \ell \end{aligned}\right\}$$ useful coordinate expressions

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Motion of particles in the potential

$\Phi = -\frac{GM}{r}$ in Newton's theory

$L = v_\varphi r$

$$E = \Phi + \frac{1}{2} v_r^2 + \frac{1}{2} v_\varphi^2$$

$$= \Phi + \frac{1}{2} v_r^2 + \frac{1}{2} \frac{L^2}{r^2}$$

$$\boxed{\frac{1}{2} v_r^2 = E - \left(\Phi + \frac{1}{2}\frac{L^2}{r^2}\right)}$$

$= \mathcal{U}_{eff}$ effective potential

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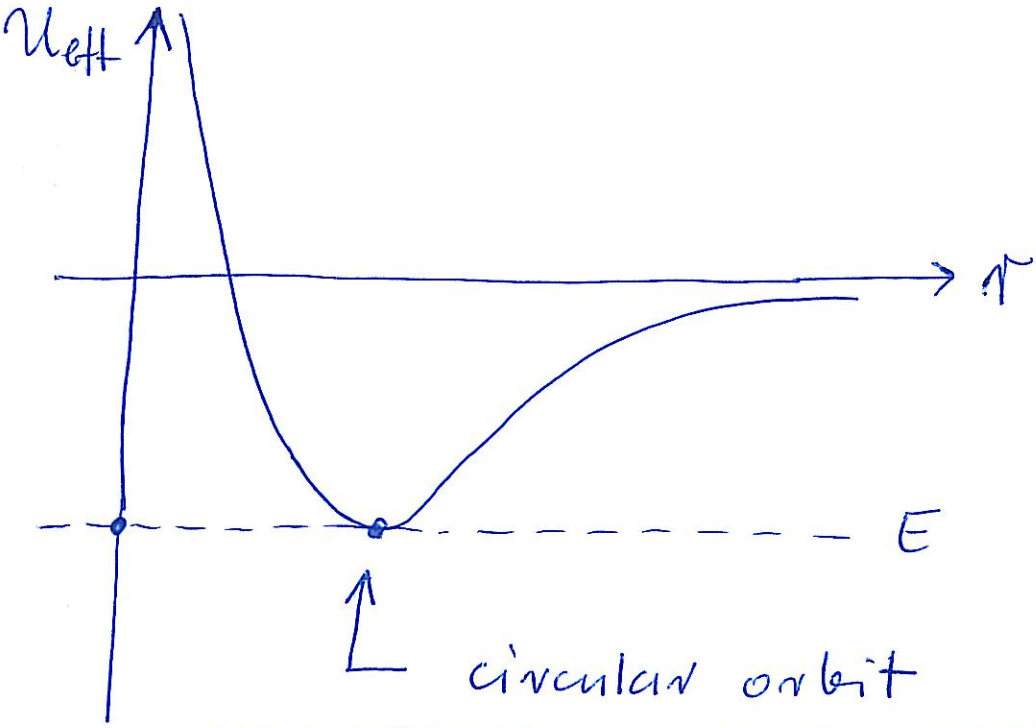
$$\boxed{\frac{1}{2} v_r^2 = E - U_{eff}}$$



$$\boxed{\begin{aligned} E &= U_{eff} \\ \left(\frac{\partial U_{eff}}{\partial r}\right)_L &= 0 \end{aligned}}$$

Slightly non-circular

$\cancel{U_{eff} = E}$

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$$\frac{1}{2}(\delta\dot{r})^2 = \delta E - \delta U_{eff}$$

$$\delta U_{eff} = \frac{1}{2}\left(\frac{\partial^2 U_{eff}}{\partial r^2}\right)_L \delta r^2$$

$$\frac{1}{2}(\delta\dot{r})^2 = \delta E - \frac{1}{2}\left(\frac{\partial^2 U_{eff}}{\partial r^2}\right)_L \delta r^2$$

take $\frac{d}{dt}$ of this:

$$(\delta\dot{r})(\delta\ddot{r}) = \delta\dot{E} - \left(\frac{\partial^2 U_{eff}}{\partial r^2}\right)_L (\delta r)(\delta\dot{r})$$

$= 0$ (as energy conserved at perturbed orbit)

$\delta\dot{r} \neq 0$

$$\boxed{\delta\ddot{r} + \left(\frac{\partial^2 U_{eff}}{\partial r^2}\right)_L \delta r = 0}$$

$$\left(\frac{\partial^2 U_{eff}}{\partial r^2}\right)_L \equiv \omega_r^2$$

epicyclic frequency

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For the $\Phi = -\frac{GM}{r}$ potential:

$$L^2 = L_K^2 \equiv GMr \quad (\text{Keplerian})$$

$$\Omega_K^2 = \frac{GM}{r^3}$$

$$\boxed{\omega_r^2 = \Omega_K^2}$$ ← closed, stable orbits

$$\omega_r^2 > 0 \Rightarrow \text{stability}$$

Circular orbits (and slightly non-circular) around black holes

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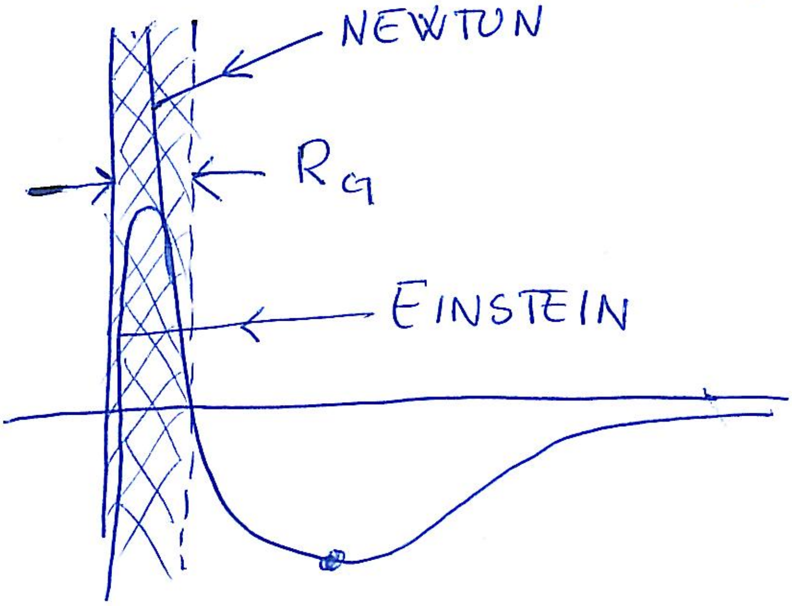
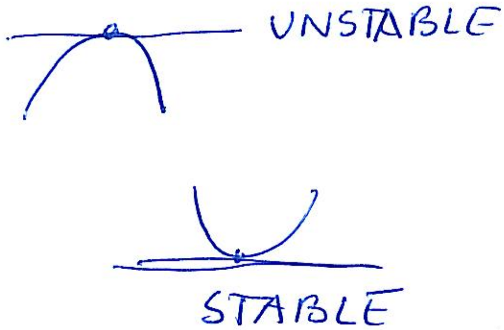
1st Wheeler's moral principle





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$$u^i = \frac{dx^i}{ds} \qquad u^t,\ u^\varphi,\ u^r$$

$$u^i = \delta^i_t u^t + \delta^i_\varphi u^\varphi + \underbrace{\delta^i_r u^r}_{= v^i}$$

$$u^i = u^t \eta^i + u^\varphi \xi^i + v^i$$

$$= u^t\left(\eta^i + \frac{u^\varphi}{u^t}\xi^i + \frac{v^i}{u^t}\right)$$

$$\frac{u^\varphi}{u^t} = \frac{d\varphi}{ds}\Big/\frac{dt}{ds} = \frac{d\varphi}{dt} = \Omega$$

angular velocity

$$\boxed{u^i = u^t\left(\eta^i + \Omega\xi^i + \tilde{v}^i\right)}$$

$\tilde{v}^i = 0$ strictly circular

$v \ll 1$ slightly non-circular

$$l = -\frac{\Omega(\xi\xi) + (\eta\xi)}{\Omega(\eta\xi) + (\eta\eta)} = -\frac{\Omega g_{\varphi\varphi} + g_{t\varphi}}{\Omega g_{t\varphi} + g_{tt}}$$

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$$\boxed{u_i u_k g^{ik} = 1}$$

$$(u_t)^2 g^{tt} + 2(u_t u_\varphi) g^{t\varphi} + (u_\varphi)^2 g^{\varphi\varphi} + (u_r)^2 g^{rr} = 1$$

$$\left[1 - (u_r)^2 g^{rr}\right] = (u_t)^2 \left[g^{tt} + 2\frac{u_\varphi}{u_t} g^{t\varphi} + \left(\frac{u_\varphi}{u_t}\right)^2 g^{\varphi\varphi}\right]$$

$$-(u_r)^2 g^{rr} = V^2 \ll 1$$

a positive (!) quantity

$$(1 + V^2) = E^2 \left[g^{tt} - 2l g^{t\varphi} + l^2 g^{\varphi\varphi}\right]$$

$$\ln(1 + V^2) = \ln E^2 + \ln\left[g^{tt} - 2l g^{t\varphi} + l^2 g^{\varphi\varphi}\right]$$

$$\frac{1}{2} V^2 = \ln E + \frac{1}{2} \ln\left[g^{tt} - 2l g^{t\varphi} + l^2 g^{\varphi\varphi}\right]$$

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$$\ln E = \mathcal{E}$$

$$\frac{1}{2} \ln \left[g^{tt} - 2\ell g^{t\varphi} + \ell^2 g^{\varphi\varphi} \right] = -\mathcal{U}_{eff}$$

$$\boxed{\frac{1}{2} V^2 = \mathcal{E} - \mathcal{U}_{eff}(r, \ell)}$$

The same equation as in Newton's theory!

The same equations have the same solutions.

Therefore:

① Circular orbits $\left(\frac{\partial \mathcal{U}_{eff}}{\partial r}\right)_\ell = 0$

② Radial epicyclic frequency $\left(\frac{\partial^2 \mathcal{U}_{eff}}{\partial r^2}\right)_\ell = \omega_r^2$

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For the Schwarzschild geometry:

$$\mathcal{U}_{eff} = -\frac{1}{2}\left[g^{tt} - 2\ell g^{t\varphi} + \ell^2 g^{\varphi\varphi}\right]$$

$$g^{tt} = \left(1 - \frac{R_G}{r}\right)^{-1}$$

$$g^{t\varphi} = 0$$

$$g^{\varphi\varphi} = -\frac{1}{r^2}$$

$$\mathcal{U}_{eff} = -\frac{1}{2}\left[\left(1-\frac{R_G}{r}\right)^{-1} - \ell^2\frac{1}{r^2}\right]$$

$$= -\frac{1}{2}\left[1 + \frac{R_G}{r} - \frac{\ell^2}{r^2}\right]$$

$$= \underset{\text{const.}}{-\frac{1}{2}} \underbrace{- \frac{GM}{r} + \frac{\ell^2}{2r^2}}_{\text{Newtonian}}$$

OK!

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$$\frac{\partial U_{eff}}{\partial r} = 0$$

$$\frac{\partial}{\partial r}\left(1-\frac{R_G}{r}\right)^{-1} = \frac{\partial}{\partial r}\left(\frac{l^2}{r^2}\right)$$

$$-\left(1-\frac{R_G}{r}\right)^{-2}\frac{R_G}{r^2} = -\frac{2l^2}{r^3}$$

$$l^2 = \frac{1}{2}R_G\, r\left(1-\frac{R_G}{r}\right)^{-2}$$

$$l^2 = GMr\left(1-\frac{R_G}{r}\right)^{-2} = GMr\, g_{tt}^{-2}$$

$$l = -\frac{\Omega g_{\varphi\varphi}}{g_{tt}}$$

$$l^2 = \frac{\Omega^2 g_{\varphi\varphi}^2}{g_{tt}^2}$$

$$\Omega^2 = \frac{1}{g_{\varphi\varphi}^2}\, l^2 g_{tt}^2 = \frac{1}{r^4}\left(l^2 g_{tt}^2\right) = \frac{GM}{r^3}$$

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$$\Omega^2 = \frac{GM}{r^3}$$

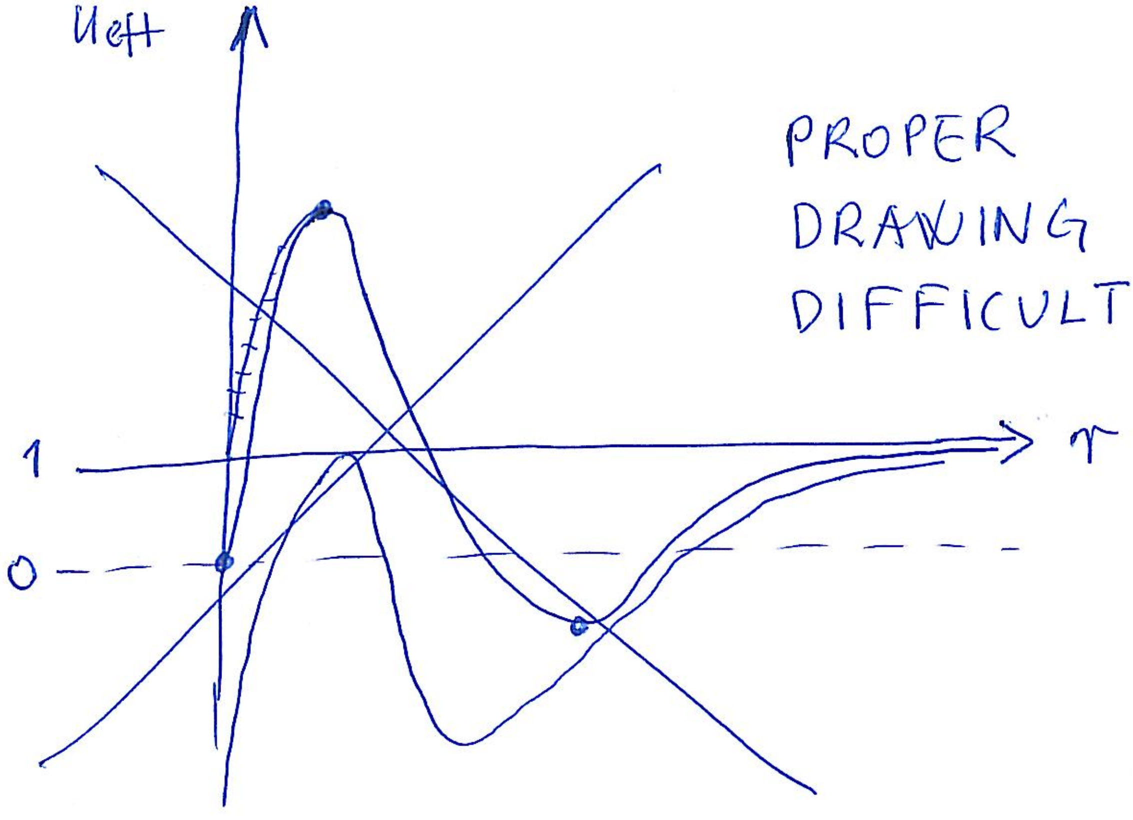
$$\omega^2 = \left(\frac{\partial^2 U_{eff}}{\partial r^2}\right)_l = \Omega^2\left(1 - \frac{r_{ms}}{r}\right)$$

$$\boxed{r_{ms} = 3R_G}$$

$$\boxed{\begin{gathered}\omega^2 < 0 \\ r \leq r_{ms}\end{gathered}}$$

The marginaly stable orbit

THE SINGLE MOST IMPORTAN FACT FOR THE BLACK HOLE ACCRETION DISK THEORY



PROPER DRAWING DIFFICULT!