

SECOND LECTURE

October 8 ①

... after adjusting to the background of the students ...

- Tasks: ① study of circular motion of particles in the Schwarzschild metric (Effective potential)
- ② derivation of the ^{TWO} most important results in the black hole accretion theory

Newton's theory (we have done this already)



$$U_{\text{eff}} = \Phi + \frac{1}{2} \frac{L^2}{r^2} ; \quad \Phi = - \frac{GM}{r}$$

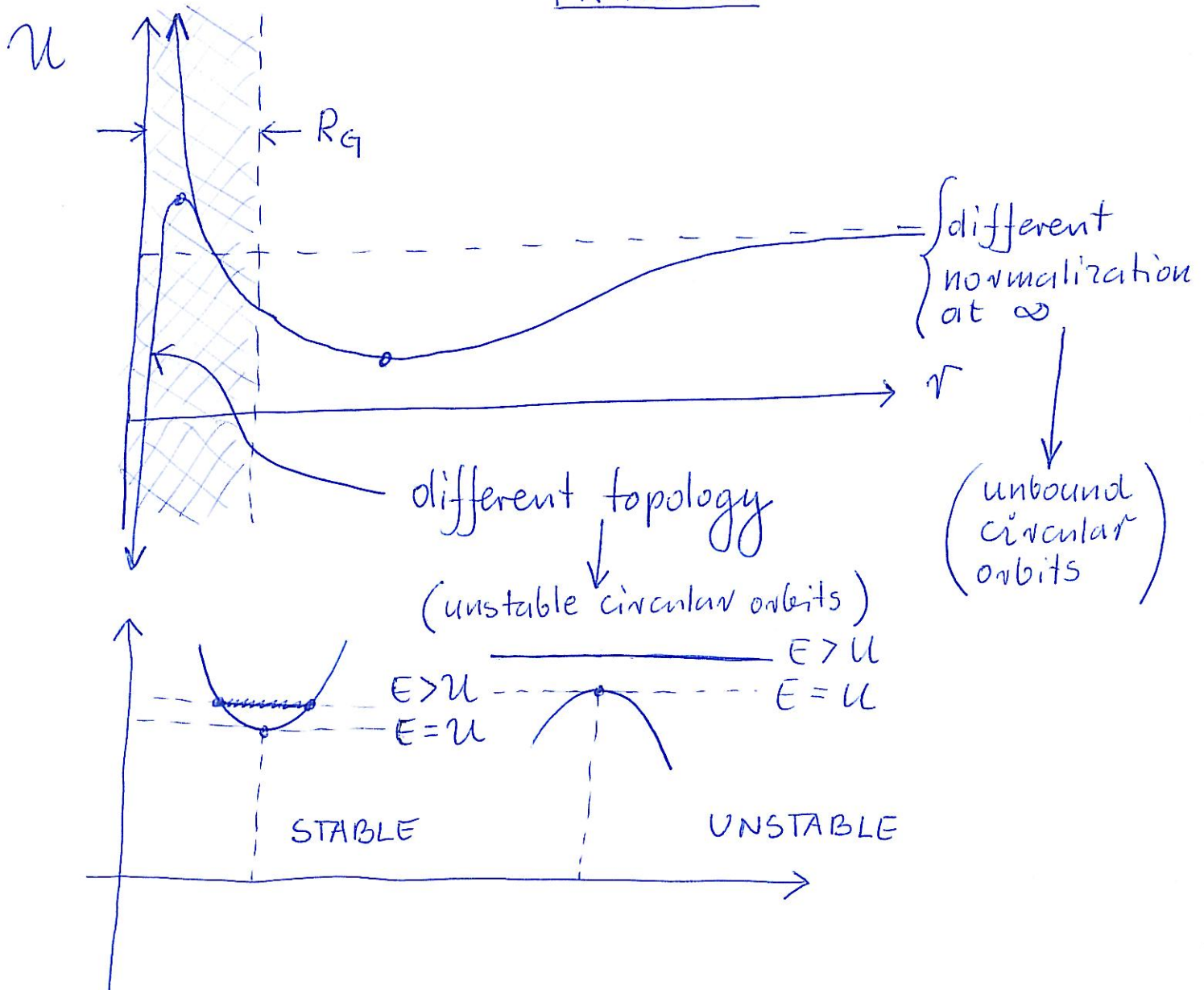
$$\left(\frac{\partial U}{\partial r} \right)_L = 0 \Rightarrow \begin{aligned} L_K^2 &= GM r \\ \Omega_K^2 &= L^2 / r^4 = \frac{GM}{r^3} \end{aligned}$$

$$\left(\frac{\partial^2 U}{\partial r^2} \right)_L = \omega_r^2 = \Omega_K^2 > 0 \text{ stable}$$

closed orbits

$$\delta \ddot{r} + \omega_r^2 \delta r = 0$$

THE FIRST WHEELER MORAL PRINCIPLE



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$$R_g \equiv \frac{2GM}{c^2}$$

$$ds^2 = \left(1 - \frac{R_g}{r}\right) c^2 dt^2 - \left(1 - \frac{R_g}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2$$

$$= g_{tt} dt^2 + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\varphi\varphi} d\varphi^2 = g_{ik} dx^i dx^k$$

KILLING VECTORS

$$\begin{cases} \partial_t g_{ik} = 0 \Rightarrow \eta^i = \delta^i_t \text{ obeys } \nabla_i \eta_k + \nabla_k \eta_i = 0 \\ \partial_\varphi g_{ik} = 0 \Rightarrow \xi^i = \delta^i_\varphi \text{ obeys } \nabla_i \xi_k + \nabla_k \xi_i = 0 \end{cases}$$

$$(\eta^i \eta_i) = (\eta\eta) = \eta^i \eta^k g_{ik} = \delta^i_t \delta^k_t g_{ik} = g_{tt}$$

$$(\eta\eta) = g_{tt} ; (\xi\xi) = g_{t\varphi} ; (\xi\eta) = 0.$$

Free motion = geodesic trajectories

$$u^i \nabla_i u_k = 0 \quad \bullet \quad \text{CONSTANT OF MOTION:}$$

Note: if $Q = u^i X_i$ and X_i is a Killing vector, then $u^k \nabla_k Q = 0$. Proof:

$$u^k \nabla_k Q = u^k \nabla_k (u^i X_i) = \cancel{u^k u^i \nabla_k X_i} + \underbrace{u^k u^i \nabla_k X_i}_{\text{geodesic}}$$

0 because $u^i u^k$ symmetric, and $\nabla_k X_i$ anti-sym.

Constant of motions here:

$$\eta^i u_i = u_t = \tilde{e} = \text{specific energy}$$

$$\xi^i u_i = u_\varphi = -\tilde{j} = \text{specific angular momentum}$$

specific = per unit mass (here particles)

Circular motion: only t and φ component — on the equatorial plane
 $\theta = \pi/2$.

$$\begin{aligned} u^i &= u^t \delta_t^i + u^\varphi \delta_\varphi^i = u^t \eta^i + u^\varphi \xi^i \\ &= u^t \left(\eta^i + \frac{u^\varphi}{u^t} \xi^i \right) = u^t \left(\eta^i + \frac{\frac{d\varphi}{ds}}{\frac{dt}{ds}} \xi^i \right) \end{aligned}$$

$$= u^t \left(\eta^i + \frac{d\varphi}{dt} \xi^i \right) = u^t (\eta^i + \Omega \xi^i)$$

$$u^i u_i = 1 = (u^t)^2 [(\eta\eta) + \Omega^2 (\xi\xi)]$$

$$\Rightarrow u^t = \frac{1}{[(\eta\eta) + \Omega^2 (\xi\xi)]^{1/2}}$$

$$u^i = \frac{1}{[(\eta\eta) + \Omega^2 (\xi\xi)]^{1/2}} (\eta^i + \Omega \xi^i)$$

Ω is a scalar independent on coordinates

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$$u_t u_t g^{tt} + u_\varphi u_\varphi g^{\varphi\varphi} = 1$$

$$2u_t \delta u_t g^{tt} + 2u_\varphi \delta u_\varphi g^{\varphi\varphi} = 0$$

$$\delta u_t = - \frac{u_\varphi}{u_t} \frac{g^{\varphi\varphi}}{g^{tt}} \delta u_\varphi$$

$= \delta e$

$$= \frac{g^{tt}}{g_{\varphi\varphi}} \delta u_\varphi = -\Omega \delta u_\varphi$$

$$= \Omega \delta \tilde{j}^*$$

$$\tilde{j} \equiv -u_\varphi$$

$$\delta e = \Omega \delta \tilde{j}$$

a very general relation:

$$dE = T dS + \Omega dJ + \mu dM$$

↑ temperature ↑ angular velocity ↑ chemical potential ↑ mass
 entropy angular momentum

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$$u_t = \tilde{e} = \text{specific energy}$$

$$-u_\varphi = \tilde{j} = \text{specific angular momentum}$$

$$l = \frac{\tilde{j}}{\tilde{e}} = \text{specific angular momentum per unit energy}$$

$$l = -\frac{u_\varphi}{u_t} = -\frac{u^\varphi g_{\varphi\varphi}}{u^t g_{tt}} = -\Omega \frac{g_{\varphi\varphi}}{g_{tt}}$$

$$l = -\Omega \frac{(\xi\xi)}{(\eta\eta)}$$

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nearly circular motion

$$u^i = u^t \eta^i + u^\varphi \xi^i + u^r \delta^i_r$$

$$1 = u^i u^k g_{ik} = (u^t)^2 g_{tt} + (u^\varphi)^2 g_{\varphi\varphi} + (u^r)^2 g_{rr}$$

$$= \frac{u_t^2}{\tilde{e}^2} g^{tt} + u_\varphi^2 g^{\varphi\varphi} + (u^r)^2 g_{rr}$$

$$1 = \boxed{\tilde{e}^2} [g^{tt} + l^2 g^{\varphi\varphi}] + (u^r)^2 g_{rr}$$

$$1 - (u^r)^2 g_{rr} = \tilde{e}^2 (g^{tt} + l^2 g^{\varphi\varphi})$$

$$\uparrow L = v^2 \quad v^2 > 0, \quad v^2 \ll 1$$

$$\ln(1 + v^2) = 2 \ln \tilde{e} + \ln(g^{tt} + l^2 g^{\varphi\varphi})$$

$$\boxed{\frac{1}{2} v^2 = \ln \underset{\substack{\parallel \\ E}}{\tilde{e}} - \mathcal{U}_{ett}(r, l^2)}$$

THE SAME
FORM AS
IN NEWTON'S
THEORY

$$\mathcal{U}_{ett} = -\frac{1}{2} \ln(g^{tt} + l^2 g^{\varphi\varphi})$$

$$\frac{1}{2} V^2 = E - U_{\text{eff}}$$

$$V^2 = (u^r)^2 g_{rr} ; \quad E = \ln \tilde{e}$$

$$U_{\text{eff}} = -\frac{1}{2} \ln (g^{tt} + l^2 g^{\varphi\varphi})$$

$$V^2 = -(u^r)^2 g_{rr} = -\left(\frac{dr}{ds}\right)^2 g_{rr} = \left(\frac{\delta \tilde{r}}{\delta s}\right)^2 = (\delta \tilde{r})^2$$

\tilde{r} = true distance in the r direction

$$U_{\text{eff}} = -\frac{1}{2} \ln (g^{tt} + l^2 g^{\varphi\varphi}) =$$

~~$$= -\frac{1}{2} \ln \left(\frac{1}{1 - \frac{R_g}{r}} \right)$$~~

~~$$= -\frac{1}{2} \ln \left(g^{tt} \left[1 + \frac{l^2}{g^{tt}} g^{\varphi\varphi} \right] \right)$$~~

~~$$= -\frac{1}{2} \ln \left(g^{tt} \left[1 - \Omega^2 \right] \right)$$~~

$$\begin{aligned}
 U_{\text{eff}} &= -\frac{1}{2} \ln \left(\frac{1}{1 - \frac{R_g}{r}} - l^2 \frac{1}{r^2} \right) \\
 &= -\frac{1}{2} \ln \left(1 + \frac{R_g}{r} - l^2 \frac{1}{r^2} \right) \\
 &= -\frac{1}{2} \left(\frac{R_g}{r} - \frac{l^2}{r^2} \right) = \frac{GM}{r} - \frac{1}{2} \frac{l^2}{r^2}
 \end{aligned}$$

OK. This is Newtonian expression

Therefore:

① ~~The~~ ~~equation~~

① The Keplerian circular orbits given by $\left(\frac{\partial U}{\partial r} \right)_e = 0$

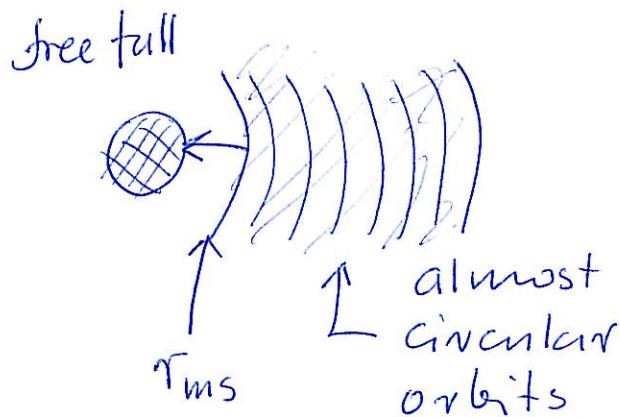
$$\Rightarrow \Omega^2 = \Omega_K^2 = \frac{GM}{r^3}$$

② Stability

$$\omega_r^2 = \left(\frac{\partial^2 U}{\partial r^2} \right)_e = \Omega_K^2 \left(1 - \frac{r_{\text{ms}}}{r} \right)$$

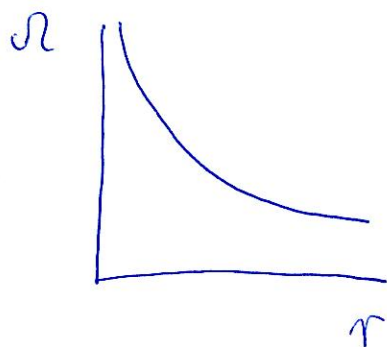
$$r_{\text{ms}} = \frac{6GM}{c^2} = 3R_g$$

"Thin disk model" of accretion:
test geodesic particles



What dissipation does?

It tries to achieve thermodynamical equilibrium with minimum energy



all mass non-rotating
near center
all angular momentum
at ∞ (with little
mass)

$$\left. \begin{array}{l} \leftarrow \delta M \\ \rightarrow \delta J \end{array} \right\}$$

general idea

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$$E = e_1 \mu_1 + e_2 \mu_2$$

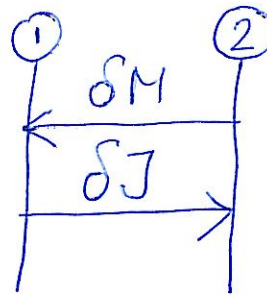
$$J = l_1 \mu_1 + l_2 \mu_2$$

$$M = \mu_1 + \mu_2$$

$$e = u_t$$

$$l = -u_\varphi$$

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$$M = m_1 + m_2$$

$$E = \tilde{e}_1 m_1 + \tilde{e}_2 m_2 \quad \tilde{e} = u_t$$

$$J = \tilde{j}_1 m_1 + \tilde{j}_2 m_2 \quad \tilde{j} = -u_\varphi$$

$$dE = \tilde{e}_1 dm_1 + \tilde{e}_2 dm_2 + m_1 d\tilde{e}_1 + m_2 d\tilde{e}_2$$

$$= \delta M (\tilde{e}_1 - \tilde{e}_2) + m_1 \frac{d\tilde{e}_1}{d\tilde{j}_1} \delta \tilde{j}_1 + m_2 \frac{d\tilde{e}_2}{d\tilde{j}_2} \delta \tilde{j}_2$$

$$\uparrow = \Omega_1 \quad \uparrow = \Omega_2$$

$$= \delta M (\tilde{e}_1 - \tilde{e}_2) + m_1 \Omega_1 \delta \tilde{j}_1 + m_2 \Omega_2 \delta \tilde{j}_2$$

$$\delta J_1 = -\delta J = \delta \tilde{j}_1 m_1 + \tilde{j}_1 \delta m_1$$

$$= \delta \tilde{j}_1 m_1 + \tilde{j}_1 \delta M$$

$$\delta \tilde{j}_1 = -\frac{1}{m_1} [\delta J + \tilde{j}_1 \delta M]$$

$$\delta J_2 = \delta J = \frac{m_2 \delta \tilde{j}_2}{\text{tongue}} - \tilde{j}_2 \delta M$$

$$\delta \tilde{j}_2 = \frac{1}{m_2} [\delta J + \tilde{j}_2 \delta M]$$

$$dE = \delta M (\tilde{e}_1 - \tilde{e}_2) - \Omega_1 [\delta J + \tilde{j}_1 \delta M] + \Omega_2 [\delta J + \tilde{j}_2 \delta M]$$

$$= \delta M [(\tilde{e}_1 - \tilde{e}_2) - \Omega_1 \tilde{j}_1 + \Omega_2 \tilde{j}_2] + \delta J [\Omega_2 - \Omega_1]$$

$$= \delta M \left[-\frac{de}{dr} dr + \frac{d(\Omega \tilde{j})}{dr} dr \right] + \delta J \left[\frac{d\Omega}{dr} dr \right]$$

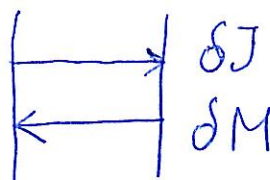
$$- \Omega \frac{d\tilde{j}}{dr} dr$$

this constant?

$$= \delta M \left[\tilde{j} \frac{d\Omega}{dr} dr \right] + \left(\frac{\delta J}{\delta M} \right) \delta M \left[\frac{d\Omega}{dr} dr \right]$$

~~$$\delta J = \tilde{j} \delta M + \Omega \delta M$$~~
~~$$\delta J = \tilde{j}_0 \delta M$$~~
~~$$\tilde{T} = 0 \text{ at } r = r_0$$~~

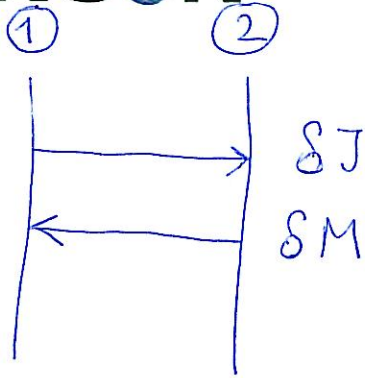
at the radius where there is no tongue



$$\delta J_0 = \tilde{j}_0 \delta M_0$$

$$- \delta J = \tilde{j}_0 \delta M$$

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$$E = E_1 + E_2 \quad E_1 = e_1 \mu_1 \quad E_2 = e_2 \mu_2$$

$$J = J_1 + J_2 \quad J_1 = l_1 \mu_1 \quad J_2 = l_2 \mu_2$$

$$\delta E = e_1 \delta M - e_2 \delta M$$

$$+ \left(\frac{\partial e}{\partial l} \right)_1 \mu_1 \delta l_1 + \left(\frac{\partial e}{\partial l} \right)_2 \mu_2 \delta l_2$$

$$\boxed{\text{crossed out}} \quad dJ_1 = d(l_1 \mu_1) = \mu_1 \delta l_1 + l_1 \delta \mu_1$$

$$- \delta J = \mu_1 \delta l_1 + l_1 \delta M \quad \parallel \quad \delta l_1 = -\frac{1}{\mu_1} (\delta J + l_1 \delta M)$$

$$+ \delta J = dJ_2 = \mu_2 \delta l_2 - l_2 \delta M$$

$$+ \delta J = \mu_2 \delta l_2 - l_2 \delta M \quad \parallel \quad \delta l_2 = \frac{1}{\mu_2} (\delta J + l_2 \delta M)$$

$$\delta E = \delta M (e_1 - e_2) + \left(\frac{\partial e}{\partial l} \right)_1 (\delta J + l_1 \delta M) + \left(\frac{\partial e}{\partial l} \right)_2 (\delta J + l_2 \delta M)$$

$$= \delta M [(e_1 - e_2) + \Omega_2 l_2 - \Omega_1 l_1] + \delta J [\Omega_2 - \Omega_1]$$

$$-\frac{de}{dl} dr + \frac{d(\Omega l)}{dr} dr$$

$$\delta \dot{E} = \delta \dot{M} \left[\frac{d\Omega}{dr} (l - l_0) \right] dr \quad \Rightarrow \quad \cancel{\Omega \frac{de}{dr} dr} + \frac{d\Omega}{dr} l dr$$

$$dE = \delta M \left(\tilde{j} \frac{d\Omega}{dr} dr \right) - \tilde{j}_0 \delta M \frac{d\Omega}{dr} dr$$

$$= \delta M \left[\frac{d\Omega}{dr} (\tilde{j} - \tilde{j}_0) \right] dr$$

unit surface: $g_{rr} dr \cdot 2\pi r$

$$(\text{flux}) = \frac{\dot{M}}{2\pi g_{rr} r} \frac{d\Omega}{dr} (\tilde{j} - \tilde{j}_0)$$

in Newton's theory $\Omega \sim \frac{1}{r^{3/2}}$
 $\tilde{j} \sim r^{1/2}$

~~$$\frac{1}{r^2} \frac{d\Omega}{dr} = \frac{r^{1/2}}{r^3} \frac{d\Omega}{dr}$$

$$= \frac{\Omega}{r^3} \frac{d\Omega}{d\ln r} = \frac{\Omega}{r^3} \frac{1}{r^{3/2+3}} = \frac{\Omega}{r^{9/2}}$$

$$\frac{\Omega}{r^3} \frac{d\Omega}{d\ln r} \sim \frac{\Omega}{r^3} \frac{1}{r^{3/2+3}} \sim \frac{\Omega}{r^{9/2}}$$~~

$$\begin{aligned}
 (\text{flux}) &\sim \frac{1}{r^2} \cdot \left(\frac{1}{r^{3/2}} \cdot \frac{1}{r} \cdot r^{1/2} \right) \\
 &\sim \frac{1}{r^2} \cdot \frac{1}{r} \cdot \frac{1}{r} \sim \frac{1}{r^3}
 \end{aligned}$$

Therefore in Newton's theory:

$$(\text{flux}) = \frac{G\dot{M}M}{2\pi r^3} \cdot \left(-\frac{3}{2}\right) \left(1 - \frac{r_0^{1/2}}{r}\right)$$

there is still $\frac{1}{2}$ because
flux goes on each side:
up - down:

$$(\text{flux}) = - \frac{3G\dot{M}M}{8\pi r^3} \left(1 - \frac{r_0^{1/2}}{r}\right)$$

$$(\text{flux}) = \dot{M} (l - l_0) \frac{d\Omega}{dr} \frac{1}{A}$$

$$(\text{power}) = L = \int (\text{flux}) A dr =$$

$$= \int \dot{M} (l - l_0) \frac{d\Omega}{dr} dr$$

$$= -\dot{M} l_0 (\Omega_\infty - \Omega_0) + \dot{M} \int l \frac{d\Omega}{dr} dr$$

$$= -\dot{M} l_0 \Omega_\infty + \dot{M} \Omega_0 l_0 +$$

$$+ \dot{M} \int \left[\frac{d}{dr} (l \Omega) - \Omega \frac{dl}{dr} \right] dr$$

$$= -\dot{M} l_0 \Omega_\infty + \dot{M} \Omega_0 l_0 +$$

$$+ \dot{M} (l_\infty \Omega_\infty - l_0 \Omega_0) - (e_\infty - e_0) \dot{M}$$

$$= -e_0 \dot{M}$$

$$L = -e_0 \dot{M} c^2$$

↑
efficiency