THIRD LECTURE

15 October (1)

$$\begin{pmatrix} \text{flux of} \\ \text{energy} \end{pmatrix} = \frac{\dot{M}}{2\pi r} \frac{d\Omega}{dr} (\tilde{j} - \tilde{j}_0)$$

in Newton's theory $\Omega \tilde{j} \sim v^2 \sim \frac{1}{r}$

therefore flux $\sim 1/r^3$ coefficient:

$\frac{3}{2} \frac{1}{2\pi} \quad \frac{3}{4\pi}$ (and $\frac{1}{2}$)

$(\text{power}) = \int \text{flux } d(\text{area}) =$

$$\frac{3GM\dot{M}}{8\pi r^3} \frac{d\ln\Omega}{d\ln r}$$

$$= \int_{r_0}^{\infty} \frac{\dot{M}}{2\pi r} \frac{d\Omega}{dr} (\tilde{j} - \tilde{j}_0) 2\pi r \, dr$$

$$= \int_{r_0}^{\infty} \dot{M} \frac{d\Omega}{dr} (\tilde{j} - \tilde{j}_0) dr = \dot{M}\Omega(\tilde{j} - \tilde{j}_0) \Big|_{r_0}^{\infty}$$

$$- \int \dot{M}\Omega \frac{d\tilde{j}}{dr} dr$$

$$= \underset{\overset{\parallel}{0}}{\dot{M}\Omega_\infty \tilde{j}_\infty} - \dot{M}\underset{\overset{\parallel}{0}}{(e_\infty} - e_0) = \dot{M} e_0$$

$$\boxed{L = e_0 \dot{M} c^2}$$

efficiency $\approx 10\%$

LAWSON

THIRD LECTURE

15 October 2

$$p = p(\varepsilon, n)$$

$$\boxed{\nabla_i T^i_k = 0} \quad \boxed{\nabla_i (n u^i) = 0}$$

Matter $\quad T^i_k = (p+\varepsilon) u^i u_k - \delta^i_k p + \tau^i_k$

stresses

For example:

$$\tau^i_k = \underbrace{f^i u_k + f_k u^i}_{\text{radiation}} + \underbrace{\eta \sigma^i_k}_{\text{viscosity}}$$

I. Stress-free case

$$T^i_k = (p+\varepsilon) u^i u_k - \delta^i_k p \; ; \quad \nabla^i (n u_i) = 0$$

$$d\varepsilon = \frac{p+\varepsilon}{n} dn + nT ds$$

$$ds = 0$$

$$d\varepsilon = \frac{p+\varepsilon}{n} dn$$

$$d(p+\varepsilon) = d\left[\frac{p+\varepsilon}{n} \cdot n\right] =$$

$$dp + d\varepsilon = \frac{p+\varepsilon}{n} dn + n\, d\left(\frac{p+\varepsilon}{n}\right)$$

$$dp + \cancel{d\varepsilon} = \cancel{d\varepsilon} + n\, d\left(\frac{p+\varepsilon}{n}\right)$$

divide side by side:

$$\frac{p+\varepsilon}{n} = w$$

$$c_s^2 = \frac{dp}{d\varepsilon} = \frac{n\, d\left(\frac{p+\varepsilon}{n}\right)}{\frac{p+\varepsilon}{n} dn} = \frac{d \ln w}{d \ln n}$$

15 October (3)

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Theorem (Bernoulli):

$$0 = u^i \nabla_i \left[\frac{p+\varepsilon}{n} (u\eta) \right]$$

Proof:

Lemma:

$$\nabla_i T^i_{\ K} \eta^K = \eta^K \underbrace{(\nabla_i T^i_{\ K})}_{=0} + \underbrace{\underbrace{T^{iK}}_{\text{symm.}} \underbrace{\nabla_i \eta_K}_{\text{anti-symm.}}}_{=0} = 0.$$

$$\nabla_i T^i_{\ K} \eta^K =$$

$$= \nabla_i \left[(p+\varepsilon) u^i (u\eta) - \eta^i p \right]$$

$$= \nabla_i \left[\frac{p+\varepsilon}{n} (u^i n)(u\eta) \right] - \underbrace{\eta^i \nabla_i p}_{=0 \text{ (assumption)}}$$

$$\Downarrow$$

$$(u^i n) \nabla_i \left[\frac{p+\varepsilon}{n} (u\eta) \right] = 0$$

q.e.d.

$$\mathcal{E} = \frac{p+\varepsilon}{n} (u\eta) \qquad \text{energy}$$

$$\mathcal{L} = -\frac{p+\varepsilon}{n} (u\xi) \qquad \text{angular momentum}$$

$$\ell = \frac{\mathcal{L}}{\mathcal{E}} = -\frac{(u\xi)}{(u\eta)}$$ also a constant of motion, as for particles.

It is a <u>kinematic</u> constant of motion, does not depend on the fluid properties.

15 October 4

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Circular orbits $u^i = A(\eta^i + \Omega \xi^i)$

Perfect fluid $T^i_{\ k} = (p+\varepsilon) u^i u_k - \delta^i_{\ k} p$

$$0 = \nabla_i T^i_{\ k} = \nabla_i \left[(p+\varepsilon) u^i u_k \right] - \nabla_k p$$

$$/\!/$$

$$(p+\varepsilon) u^i \nabla_i u_k - \nabla_k p = 0$$

$$\boxed{a_k \equiv u^i \nabla_i u_k = \frac{\nabla_k p}{p+\varepsilon}}$$

$$a_k = \left[A(\eta^i + \Omega \xi^i) \right] \nabla_i \left[A(\eta_k + \Omega \xi_k) \right]$$

$$= A^2 \left[(\eta^i \nabla_i \eta_k) + 2\Omega (\eta^i \nabla_i \xi_k) + \Omega^2 (\xi^i \nabla_i \xi_k) \right]$$

$$= -\frac{1}{2} A^2 \left[\nabla_k (\eta\eta) + \Omega^2 \nabla_k (\xi\xi) \right]$$

$$= -\frac{1}{2} \frac{\nabla_k (\eta\eta) + \Omega^2 \nabla_k (\xi\xi)}{(\eta\eta) + \Omega^2 (\xi\xi)} =$$

$$= -\frac{1}{2} \frac{\nabla_k \left[(\eta\eta) + \Omega^2 \nabla_k (\xi\xi) \right] - (\xi\xi) \nabla_k \Omega^2}{(\eta\eta) + \Omega^2 (\xi\xi)}$$

$$= -\frac{1}{2} \nabla_k \ln A^{-2} + \frac{(\xi\xi) \Omega \nabla_k \Omega^2}{(\eta\eta) \left[1 + \Omega^2 \frac{(\xi\xi)}{(\eta\eta)} \right]}$$

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$$= \nabla_k \ln A - \frac{l \nabla_k \Omega}{1 - \Omega l}$$

$$\boxed{\frac{\nabla_k p}{p + \varepsilon} = \nabla_k \ln A - \frac{l \nabla_k \Omega}{1 - \Omega l}}$$

von Zeipel: if $p = p(\varepsilon)$ then $l = l(\Omega)$!

Important case $l = \text{const}$.

$$\nabla_k \ln A - \frac{l \nabla_k \Omega}{1 - \Omega l} = \nabla_k \ln A + \frac{\nabla_k (1 - \Omega l)}{1 - \Omega l}$$

$$= \nabla_k \ln \left[A \cdot (1 - \Omega l) \right]$$

$$A \cdot (1 - \Omega l) = u^t \left(1 + \frac{u_\varphi u^\varphi}{u_t u^t} \right) =$$

$$= u^t \frac{u^t u_t + u^\varphi u_\varphi}{u_t u^t} = \frac{u^t}{u^t u_t} = \frac{1}{u_t}$$

$$\boxed{\frac{\nabla_k p}{p + \varepsilon} = -\frac{1}{2} \nabla_k \ln u_t^2} \quad l = \text{const}$$

$$u_t^2 = (u^t g_{tt})^2 = \frac{g_{tt}^2}{g_{tt} + \Omega^2 (g_{\varphi\varphi})}$$

THIRD LECTURE

$$u_t^2 = \frac{g_{tt}^2}{g_{tt} + \ell^2 \left(\frac{g_{tt}}{g_{\varphi\varphi}}\right)^2 g_{\varphi\varphi}} = \frac{g_{tt}}{1 + \ell^2 g_{tt} g_{\varphi\varphi}^{-1}}$$

$\ell^2/r^2 \sim v^2$ OK.

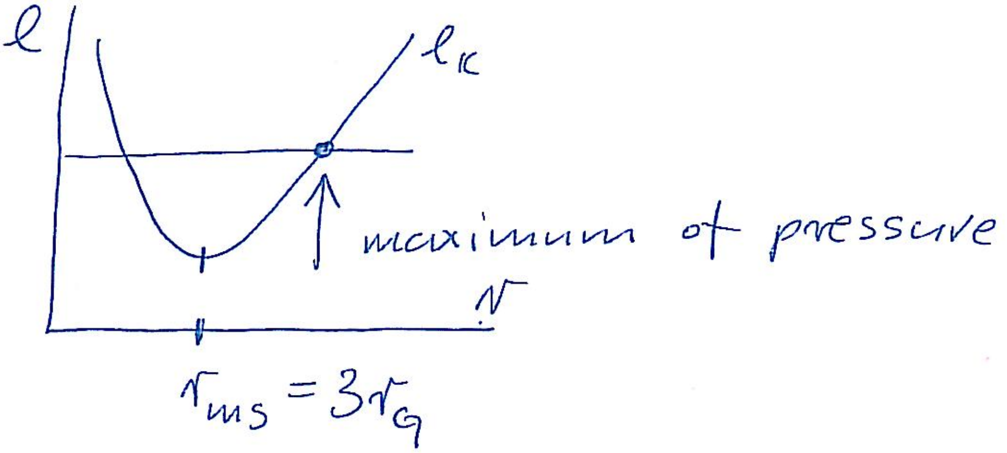
$$\boxed{u_t^2 = \frac{g_{tt}}{1 + \ell^2 \frac{g_{tt}}{g_{\varphi\varphi}}}}$$

$$u_t^2 = f(r, \theta; \ell) = \text{const}$$

coincide with $p(r,\theta) = \text{const}$.

<u>Important note:</u>

$$\nabla_K p = 0 \implies a_K = 0 \implies \ell = \ell_K$$



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cases

$\ell < \ell_{ms}$

$\ell = \ell_{ms}$

$\ell < \ell_{mb}$

$\ell = \ell_{mb}$

$\ell > \ell_{mb}$

The second crossing and the Roche-lobe overflow (story)

The sonic point

Newton:

$$u^i \nabla_i E = 0 \Rightarrow \left(\Phi + \frac{1}{2}\frac{\ell^2}{r^2}\right)' + \left(\frac{1}{2}v^2\right)' + \left(\int \frac{dp}{\rho}\right)' = 0$$

$$\boxed{\mathcal{U}' + vv' + \frac{p'}{\rho} = 0}$$

$$\vec{\nabla}(\vec{u}\cdot\rho) = 0 = \boxed{\frac{v'}{v} + \frac{\rho'}{\rho} = 0}$$

eleminate $\frac{v'}{v}$

kind of → $$\boxed{\frac{\rho'}{\rho} = \frac{\mathcal{U}' - \frac{2}{r}}{(c_s^2 - v^2)}}$$

$$\frac{\mathcal{U}'}{v^2} + \frac{v'}{v} + \frac{c_s^2}{v^2}\frac{\rho'}{\rho} = 0$$

$$-\frac{v'}{v} - \frac{\rho'}{\rho} = 0$$ ← 2/r missing

$$\Rightarrow \frac{\rho'}{\rho}\left(\frac{c_s^2}{v^2} - 1\right) = -\frac{\mathcal{U}'}{v^2} - 2/r$$

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conical flow

$u^\theta \equiv 0$

15 October (8)

$$u^i \nabla_i E = 0$$

$$u^r \partial_r \left(\frac{p+\varepsilon}{n} u_t \right) = 0$$

$$\frac{w'}{w} = -\frac{u_t'}{u_t}$$

$$\frac{w'}{w} = -\frac{u_\varphi'}{u_\varphi}$$

$$\frac{w'}{w} = c_s^2 \frac{n'}{n}$$

$$g^{tt} u_t^2 c_s^2 \frac{n'}{n} = -u_t' u_t g^{tt} = -\frac{1}{2} (u_t^2)' g^{tt}$$

$$g^{tt} u_t^2 c_s^2 \frac{n'}{n} = -\frac{1}{2} \left(u_t^2 g^{tt} \right)' + \frac{1}{2} u_t^2 (g^{tt})'$$

$$g^{\varphi\varphi} u_\varphi^2 c_s^2 \frac{n'}{n} = -\frac{1}{2} \left(u_\varphi^2 g^{\varphi\varphi} \right)' + \frac{1}{2} u_\varphi^2 (g^{\varphi\varphi})'$$

$+$

$$(1 - g^{rr} u_r^2) c_s^2 \frac{n'}{n} = -\frac{1}{2} (1 - g^{rr} u_r^2)' + \frac{1}{2} u_t^2 (e^{-2U})'$$

$$-g^{rr} u_r^2 \equiv V^2$$

$$(1+V^2) c_s^2 \frac{n'}{n} = -\frac{1}{2} (1+V^2)' - \frac{1}{2} u_t^2 (e^{-2U})'$$

$$(1+V^2) c_s^2 \frac{n'}{n} = -VV' - \frac{1}{2} u_t^2 (e^{-2U})'$$

$$\left[\frac{(1+V^2) c_s^2}{V^2} \right] \frac{n'}{n} = -\frac{V'}{V} - \frac{1}{2} \frac{u_t^2}{V^2} (e^{-2U})'$$

$$0 = \nabla_i (r^2 n g^{rr} u_r) = \frac{2}{r} + \frac{n'}{n} + \frac{u_r'}{u_r} + \frac{(g^{rr})'}{g^{rr}}$$

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15 October

$$2\frac{V'}{V} = \frac{\left(u_r^2 g^{rr}\right)'}{u_r^2 g^{rr}} = 2\frac{u_r'}{u_r} + \frac{\left(g^{rr}\right)'}{g^{rr}}$$

$$\frac{u_r'}{u_r} = \frac{V'}{V} - \frac{1}{2}\frac{\left(g^{rr}\right)'}{g^{rr}}$$

$$0 = \frac{2}{r} + \frac{n'}{n} + \frac{V'}{V} + \frac{1}{2}\frac{\left(g^{rr}\right)'}{g^{rr}}$$

$$\left[\frac{(1+V^2)c_s^2}{V^2}\right]\frac{n'}{n} = -\frac{V'}{V} + \frac{1}{2}\frac{u_t^2}{V^2}\left(e^{-2U}\right)'$$

$$\frac{n'}{n}\left[\frac{(1+V^2)c_s^2}{V^2} - 1\right] = \frac{1}{2}\frac{u_t^2}{V^2}\left(e^{-2U}\right)' + \frac{2}{r} + \frac{1}{2}\frac{\left(g^{rr}\right)'}{g^{rr}}$$

$\Uparrow$ sonic point

$$(1+V^2)c_s^2 - V^2 = 0$$

$$c_s^2 = \frac{V^2}{1+V^2}$$

$$c_s^2 + V^2\left(c_s^2 - 1\right) = 0$$

$$V^2 = \frac{c_s^2}{1-c_s^2}$$