FLUID EQUATIONS AS MOMENTS OF THE BOLTZMANN EQUATIONS

Let us consider the Boltzmann equation in the form

$$\frac{\partial f}{\partial t} + \vec{v}\,\frac{\partial f}{\partial \vec{x}} + \vec{F}\,\frac{\partial f}{\partial \vec{v}} = \left(\frac{\partial f}{\partial t}\right)_c \tag{1}$$

Let he distribution function has the form

$$f_\alpha = f_\alpha(\vec{x}, \vec{v}, t) \tag{2}$$

where $\vec{x}$ labels the coordinate vector, $\vec{v}$ the velocity vector and α the sort of particles (electrons, ions).

Let us define the following moments:

$$n_\alpha = \int f_\alpha(\vec{x}, \vec{v}, t)\, d\vec{v}' \tag{3}$$

$$\vec{v}_\alpha = \frac{1}{n_\alpha} \int f_\alpha(\vec{x}, \vec{v}'\, t)\,\vec{v}'\, d\vec{v}' \tag{4}$$

$$\vec{\vec{P}}_\alpha = m_\alpha \int f_\alpha(\vec{x}, \vec{v}', t)(\vec{v}' - \vec{v}_\alpha)\, d\vec{v}'. \tag{5}$$

Here, $n_\alpha, \vec{v}_\alpha, \vec{\vec{P}}_\alpha$ are the density, the averaged velocity and the stress (pressure) tensor of the αth component, respectively.

For the isotropic distribution, the pressure is scalar

$$p_\alpha = \frac{1}{3} \int f_\alpha(\vec{x}, \vec{v}', t)(\vec{v}' - \vec{v}_\alpha)^2\, d\vec{v}'. \tag{6}$$

The moments of the Boltzmann equation (the index "α" is leaving out) will then be used for the derivation of the equation of continuity and of the fluid equation of motion.

1. The continuity equation – the lowest moment – is derived by means of the integration of the Boltzmann euation, as follows:

$$\int \frac{\partial f}{\partial t}\, d\vec{v} + \int \vec{v}\,\frac{\partial f}{\partial \vec{x}}\, d\vec{v} + \frac{q}{m}\int \left(\vec{E} + \vec{v}\times\vec{B}\right)\frac{\partial f}{\partial \vec{v}}\, d\vec{v} = \int \left(\frac{\partial f}{\partial t}\right)_c d\vec{v}\,, \qquad (7)$$

where for the force F we choose the Lorentz force

$$F = q\left(\vec{E} + \vec{v}\times\vec{B}\right). \qquad (8)$$

The first term of the foregoig equation gives:

$$\int \frac{\partial f}{\partial t}\, d\vec{v} = \frac{\partial}{\partial t}\int f\, d\vec{v} = \frac{\partial n}{\partial t}\,. \qquad (9)$$

Let us consider the term $\int \vec{v}\,\nabla f\, d\vec{v}$. Here, $\vec{v}$ is an independent variavbe (together with $\vec{x}$) and from this follows that $\vec{v}$ is not affected by the gradient ∇. Consequently,

$$\int \vec{v}\cdot\nabla f\, d\vec{v} = \nabla\cdot\int \vec{v}\, f\, d\vec{v} = \nabla\cdot\left(n\,\bar{\vec{v}}\right) = \nabla\cdot\left(n\,\vec{u}\right) \qquad (10)$$

where $\bar{\vec{v}} = \vec{u}$ is the averaged velocity.

Since according to assumption f vanishes faster than $\frac{1}{v^2}$, the integral over $\vec{E},\ \ \vec{v}\times\vec{B}$ vanishes also.

Since collisions cannot change the total number of particles, also the integral over $\left(\frac{\partial f}{\partial t}\right)_c$ vanishes. Consequently, the zero momentum of the Boltzmann equation reeds:

$$\frac{\partial n}{\partial t} + \nabla\cdot\left(n\,\vec{u}\right) = 0 \qquad (11)$$

and gives the continuity equation.

The next moment, which derives the equation of motion, is obtained by multiplyting the Boltzmann equation by $m\,\vec{v}$ and integrating over $d\,\vec{v}$. This form then reeds:

$$m\int \vec{v}\,\frac{\partial f}{\partial t}\, d\,\vec{v} + m\int \vec{v}\left(\vec{v}\cdot\nabla f\right) d\,\vec{v} + q\int \vec{v}\left(\vec{E} + \vec{v}\times\vec{B}\right)\frac{\partial f}{\partial \vec{v}}\, d\,\vec{v} = \int m\,\vec{v}\left(\frac{\partial f}{dt}\right)_c d\,\vec{v}\ . \qquad (12)$$

Let us now express all terms of the foregoing equation

$$m\int \vec{v}\,\frac{\partial f}{\partial t}\,d\,\vec{v} = m\frac{\partial}{\partial t}\int \vec{v}\,f\,d\,\vec{v} = m\frac{\partial}{\partial t}\left(n\,\vec{u}\right) \tag{13}$$

$$q\int \vec{v}\left(\vec{E}+\vec{v}\times\vec{B}\right)\frac{\partial f}{\partial \vec{v}}\,d\,\vec{v} = -q\int f\,\frac{\partial}{\partial \vec{v}}\cdot\left[\vec{v}\left(\vec{E}+\vec{v}\times\vec{B}\right)\right]d\,\vec{v} = -q\,n\left(\vec{E}+\vec{u}\times\vec{B}\right), \tag{14}$$

where we have used the assumed property of the distribution function $\lim_{v\to\infty} f(v)\to 0$ and the property

$$\frac{\partial}{\partial \vec{v}}\cdot\left[\vec{E}+\vec{v}\times\vec{B}\right]=0. \tag{15}$$

Further, the last term gives

$$m\int \nabla\cdot\left(f\,\vec{v}\,\vec{v}\right)d\,\vec{v} = m\,\nabla\cdot\int f\,\vec{v}\,\vec{v}\,d\,\vec{v}\,. \tag{16}$$

Expressing

$$\vec{v} = \vec{u} + \vec{w} \tag{17}$$

where $\vec{u}$ represents the averaged velocity and $\vec{w}$ the thermal velocity, and using the identity

$$\nabla\cdot\left(n\,\vec{u}\,\vec{u}\right) = \vec{u}\,\nabla\cdot\left(n\,\vec{u}\right) + n\left(\vec{u}\cdot\nabla\right)\vec{u}\,, \tag{19}$$

the term $m\int \nabla\cdot\left(f\,\vec{v}\,\vec{v}\right)d\,\vec{v}$ can be expressed as

$$m\int \nabla\cdot\left(f\,\vec{v}\,\vec{v}\right)d\,\vec{v} \;=\; m\,\vec{u}\,\nabla\cdot\left(n\,\vec{u}\right) + m\,n\left(\vec{u}\cdot\nabla\right)\vec{u} + \nabla\cdot\vec{\vec{P}}\,. \tag{20}$$

Here, $\vec{\vec{P}} = m\,n\,\vec{w}\,\vec{w}$ is the stress tensor. Further, we have used the following identity (e.g., in Cartesian coordinates)

$$\left(\vec{u}\cdot\nabla\right)\vec{G} = u_x\,\frac{\partial \vec{G}}{\partial x} + u_y\,\frac{\partial \vec{G}}{\partial y} + u_z\,\frac{\partial \vec{G}}{dz}\,. \tag{21}$$

Expressing further the colision term $\vec{P}_{i,j}$ in the form

$$\int m\,\vec{v}\left(\frac{\partial f}{\partial t}\right)_c d\,\vec{v} \;=\; \vec{P}_{i,j} = m\,n\left(\vec{v}_i - \vec{v}_j\right)\nu_{i,j} \tag{22}$$

where $\nu_{i,j}$ is the collision frequency between i_{th} and j_{th} fluid component (e.g. between electrons and ions) and inserting the terms (13), (14), (16), (20) and (22) into (13), we obtain the equation of motion on the following form:

$$m\,n\left[\frac{\partial \vec{u}}{\partial t}+(\vec{u}\cdot\nabla)\vec{u}\right]=q\,n\left(\vec{E}+\vec{u}\times\vec{B}\right)-\nabla\cdot\vec{\vec{P}}+\vec{P}_{i,j} \tag{23}$$

For the isotropic case, $\nabla\cdot\vec{\vec{P}}=\nabla\,p$.

The complete set of fluid equations must be closed by the Maxwell equations and by the equation of state and form the following system:

$$\varepsilon_0\,\nabla\cdot\vec{E}=n_i\,q_i+n_e\,q_e$$

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\cdot\vec{B}=0$$

$$\nabla\times\vec{B}=\mu_0\left[n_i\,q_i\,\vec{u}_i+n_e\,q_e\,\vec{u}_e+\varepsilon_0\,\frac{\partial\vec{E}}{\partial t}\right]$$

$$\frac{\partial n_j}{\partial t}+\nabla\cdot\left(n_j\,\vec{u}_j\right)=0$$

$$m_j\,n_j\left[\frac{\partial\vec{u}_j}{\partial t}+(\vec{u}_j\cdot\nabla)\vec{u}_j\right]=q_j\,n_j\left(\vec{E}+\vec{u}_j\times\vec{B}\right)-\nabla p_j$$

$$p_j=C_j\,n_j^{\gamma_j};\ \gamma_j=\frac{C_{pj}}{C_{vj}}$$

This forms 18 scalar equations for $n_i, n_e, p_{i,}\ p_{e,}\ \vec{u}_i, \vec{u}_e, \vec{E},\ \vec{B}$.

THE SINGLE-FLUID MAGNETOHYDRODYNAMIC (MHD) EQUATIONS

In the MHD model, the separate identities of electrons and ions do not appear. Instead, the mass density ρ, mass velocity $\vec{v}$, current density $\vec{j}$ and charge density σ are defined as follows:

$$\rho = n_i\, M + n_e\, m \approx n\left(M + m\right)$$

$$\vec{v} = \frac{1}{\rho}\left(n_i\, M\, v_i + n_e\, m\, v_e\right) = \frac{M\,\vec{v}_i + m\,\vec{v}_e}{M + m}$$

$$\vec{j} = e\left(n_i\,\vec{v}_i - n_e\,\vec{v}_e\right) \cong n\, e\left(\vec{v}_i - \vec{v}_e\right).$$

$$\sigma = n_i\, q_i + n_e\, q_e\,.$$

Here, e is the electron charge, n_i , M , $\vec{v}_i$ and n_e , m , $\vec{v}_e$ are density, mass and averaged velocity of the ion and electron component, respectively.

Let us consider the fluid equation of motion for the ion and electron component:

$$M\, n\frac{\partial\,\vec{v}_i}{\partial\, t} = e\, n\left(\vec{E} + \vec{v}_i \times \vec{B}\right) - \nabla\, p_i + \vec{P}_{ie} \tag{24}$$

$$m\, n\frac{\partial\,\vec{v}_e}{\partial\, t} = -\,e\, n\left(\vec{E} + \vec{v}_e \times \vec{B}\right) - \nabla\, p_e + \vec{P}_{ei}\,. \tag{25}$$

Here, $\vec{P}_{ei} \approx m\, n_e\left(\vec{v}_i - \vec{v}_e\right)\nu_{ei}$, where ν_{ei} is the collision frequency.

We now add equations (24), (25), obtaining

$$n\frac{\partial}{\partial t}\left(M\,\vec{v}_i + m\,\vec{v}_e\right) = e\,n\left(\vec{v}_i - \vec{v}_e\right)\times\vec{B} - \nabla\,p$$

since $\vec{P}_{ie} = -\,\vec{P}_{ei}$ and $p = p_i + p_e$.

Using our definition for the mass density, velocity and current, we obtain from the foregoing equation the MHD motion equation

$$\rho\frac{\partial\,\vec{v}}{\partial\,t} = \vec{j}\times\vec{B} - \nabla\,p. \qquad (26)$$

The equation of continuity for mass ρ can be simly obtained from the equations for continuity of electron and ion components, multiplying them by the ion and electron masses, respectively and added. The resulting MHD continuity equation then reads:

$$\frac{\partial\,\rho}{\partial\,t} + \nabla\cdot\left(\rho\,\vec{v}\right) = 0. \qquad (27)$$

We shall now derive the generalized Ohm's law. Let us multiply Eq. (24) by m and Eq.(25) by M and substract the latter from the former. This will result in the following equation

$$M\,m\,n\frac{\partial}{\partial t}\left(\vec{v}_i - \vec{v}_e\right) = e\,n\left(M+m\right)\vec{E} + e\,n\left(m\,\vec{v}_i + M\,\vec{v}_e\right)\times\vec{B} - m\,\nabla\,p_i + M\,\nabla\,p_e - \left(M+m\right)\vec{P}_{ei}$$

Let us express the collision term $\vec{P}_{ei}$ as follows

$$\vec{P}_{ei} = \eta\,e^2\,n^2\left(\vec{v}_i - \vec{v}_e\right).$$

Here, η is the specific resistivity.

Then, using our definitions for the mass density, velocity and current, we obtain

$$\frac{M\,m\,n}{e}\frac{\partial}{\partial\,t}\left(\frac{\vec{j}}{n}\right) = e\,\rho\,\vec{E} - \left(M+m\right)n\,e\,\eta\,\vec{j} - m\,\nabla\,p_i + M\,\nabla\,p_e + e\,n\left(m\,\vec{v}_i + M\,\vec{v}_e\right)\times\vec{B}. \qquad (28)$$

Using further the following expression

$$m\,\vec{v}_i + M\,\vec{v}_e = M\,\vec{v}_i + m\,\vec{v}_e + M\left(\vec{v}_e - \vec{v}_i\right) + m\left(\vec{v}_i - \vec{v}_e\right) = \frac{\rho}{n}\vec{v} - (M - m)\frac{\vec{j}}{n_e}$$

and dividing the equation (28) by $e\rho$, neglecting $m \ll M$ and considering $\frac{\partial}{\partial t} \approx 0$, we finally obtain the generalized Ohm's law

$$\vec{E} + \vec{v} \times \vec{B} = \eta\,\vec{j} + \frac{1}{e\,n}\left(\vec{j} \times \vec{B} - \nabla\, p_e\right) \tag{29}$$

where, usually, the last term can be neglected.

Consequently, the system of MHD equations is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho\,\vec{v}\right) = 0$$

$$\rho \frac{\partial \vec{v}}{\partial t} = \vec{j} \times \vec{B} - \nabla\, p.$$

$$\vec{E} + \vec{v} \times \vec{B} = \eta\,\vec{j} + \frac{1}{e\,n}\left(\vec{j} \times \vec{B} - \nabla\, p_e\right)$$

which must be completed by the Maxwell equations

$$\varepsilon_0\, \nabla \cdot \vec{E} = \sigma$$

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}$$

$$\nabla \cdot \vec{B} = 0$$

$$\frac{1}{\mu_0} \nabla \times \vec{B} = \vec{j}\,.$$