WAVES IN PLASMAS

Waves in plasmas appear as the consequence of an instability, or as an consequence of an artificial excitation, either for heating, or for diagnostics.

There exists a broad spectrum of waves, which differ in frequencies, wave numbers, polarization and absorption.

The wave propagation is basically different in a plasma without and with magnetic field.

Typical wave frequencies in plasmas are in the region $1\ MHz - 10^2\ GHz$.

Any sinusoidal oscillating quantity – e.g. the density n can be represented as follows:

$$n = \bar{n} \exp\left[i\left(\vec{k} \cdot \vec{r} - \omega t\right)\right].$$

Here, e.g., in Cartesian coordinates,

$$\vec{k} \cdot \vec{r} = k_x x + k_y y + k_z z .$$

In discussion of wave propagation, two velocities are usually mentioned, the phase velocity, v_ϕ, and the qroup velocity, v_g.

The phase velocity is defined as

$$v_\phi = \frac{\omega}{k}$$

whereas the group velocity is defined as

$$v_g = \frac{\partial \omega}{\partial k}.$$

The final aim of the study of wave-propagation properties consist in finding of the corresponding dispersion relation, which couples possible combination of the wave frequency ω and of the wave vector $\vec{k}$, i.e. , of the relation

$$D = D\left(\omega, \vec{k}\right) = 0.$$

Plasma oscillation

If the electrons in a plasma are displaced from a uniform background of ions, electric field appears in a such direction as to restore the neutrality of the plasma pulling the electrons back to their initial position. Because of their inertia, the electrons will overshoot and oscillate around their equilibrium position with the characteristic frequency, called *plasma frequency* (Chen). We shall now derive this frequency.

Let us assume the following simplification:

1. There is no magnetic field.
2. There is no thermal motion (plasma is cold).
3. The ions are fixed in space in an uniform distribution.
4. The electron motion occurs only in the x direction.

From this follows: $\nabla = \vec{x}\,\dfrac{\partial}{\partial x}$, $\quad \vec{E} = E\,\vec{x}^{\,0}$, $\nabla \times \vec{E} = 0$; $\vec{E} = -grad\phi$.

In this case, the equation of motion ot the electron fluid component is

$$m\,n_e\left[\frac{\partial\,\vec{v}_e}{\partial\,t} + \left(\vec{v}_e\cdot\nabla\right)\vec{v}_e\right] = -e\,n_e\,\vec{E}. \tag{1}$$

The corresponding equation of continuity of the electron fluid component is

$$\frac{\partial\,n_e}{\partial\,t} + \nabla\cdot\left(n_e\,\vec{v}_e\right) = 0. \tag{2}$$

From the Maxwell equations, only the Poisson equation remains:

$$\varepsilon_0\,\nabla\cdot\vec{E} = \varepsilon_0\,\frac{\partial\,\vec{E}}{\partial\,x} = e\left(n_i - n_e\right). \tag{3}$$

The foregoing equations will be solved by the *linearization.* Let:

$$n = n_0 + n_1; \quad \vec{v}_e = \vec{v}_0 + \vec{v}_1; \quad \vec{E} = \vec{E}_0 + \vec{E}_1 \text{ and let } \frac{n_1}{n_0} << 1.$$

Further, we choose:

$$\nabla\,n_0 = 0, \quad \vec{v}_0 = 0, \quad \vec{E}_0 = 0, \quad \frac{\partial\,n_0}{\partial\,t} = 0.$$

Since the term $\left(\vec{v}_1\cdot\nabla\right)\vec{v}_1$ is of the second order, the equation of motion is:

$$m\left[\frac{\partial\,\vec{v}_1}{\partial\,t}\right] = -e\,\vec{E}_1,$$

the equation of continuity is (neglecting again the second order quantity $n_1\,\vec{v}_1 << 1$ and using the assumption $\nabla\,n_0 = 0$)

$$\frac{\partial n_1}{\partial t} + n_0 \nabla \cdot \vec{v}_1 = 0,$$

and the Poisson equation is

$$\varepsilon_0 \nabla \cdot \vec{E}_1 = -e\, n_1 .$$

Let us use the following representation:

$$\vec{v}_1 = \vec{v}_{10}\, e^{i(kx-\omega t)} \cdot \vec{x}_0; \qquad n_1 = n_{10}\, e^{i(kx-\omega t)}; \qquad \vec{E}_1 = \vec{E}_{10}\, e^{i(kx-\omega t)}. \tag{4}$$

Then the time derivative $\frac{\partial}{\partial t}$ can be repklaced by $-i\omega$ and the gradient ∇ **by** $\boldsymbol{i\, k\, \vec{x}_0}$.

Equations (1), (2), (3) can be replaced by a simple system of algebraic equations:

$$-i\, m\, \omega\, v_1 = -e\, E_1 \tag{5}$$

$$-i\, \omega\, n_1 = -n_0\, i\, k\, v_1 \tag{6}$$

$$i\, k\, \varepsilon_0\, E_1 = -e\, n_1 \tag{7}$$

Eliminating n_1, v_1, we obtain the equation

$$-i\, m\, \omega\, v_1 = -i \frac{n_0 e^2}{\varepsilon_0 \omega} v_1,$$

which gives the solution

$$\omega^2 = \frac{n_0 e^2}{\varepsilon_0 m} = \omega_{pe}^{\ 2}.$$

Here, $\omega_{pe} = \left(\frac{n_0 e^2}{\varepsilon_0 m} \right)^{\frac{1}{2}}$ is the *electron plasma frequency.*

Electron plasma waves.

In the foregoing equation of motion, we have neglected the effect of the pressure. Now ,we shall consider the equation of motion in the full form

$$m\, n_e \left[\frac{\partial \vec{v}_e}{\partial t} + (\vec{v}_e \cdot \nabla)\, \vec{v}_e \right] = -e\, n_e\, \vec{E} - \nabla p_e$$

Let us consider $\frac{\partial T_e}{\partial x} = 0$. Then, the gradient of the pressure can be expressed in the following form:

$$\nabla p_e = \nabla (\gamma\, n_e\, K\, T_e) = K\, T_e\, \gamma\, \nabla n_e$$

Here, K is the Boltzmann constant, and γ is the ratio of specific heats

$$\gamma = C_p / C_v .$$

It appears in the thermodynamic equation of state and relates p to n according to the expression

$$p = C\, \rho^{\gamma}$$

where ρ is the mass density.

If N is the number of degrees of freedom, then for the adiabatic compression

$$\gamma = \frac{1}{N}(2 + N).$$

For one-dimensional case, $N = 1$ and therefore $\gamma = 3$. Then,

$$\nabla p_e = 3\, K\, T_e\, \frac{\partial n_e}{\partial x}\, \vec{x}_0 .$$

Consequently, the equation of motion takes the form

$$m\, n_e \left[\frac{\partial \vec{v}_e}{\partial t} + (\vec{v}_e \cdot \nabla)\, \vec{v}_e \right] = -e\, n_e\, \vec{E} - 3\, K\, T_e\, \frac{\partial n_e}{\partial x} .$$

Using again the linearization, we obtain

$$m\, n_0 \frac{\partial \vec{v}_1}{\partial t} = -e\, n_0\, \vec{E}_1 - 3\, K\, T_e \frac{\partial n_1}{\partial x}.$$

Using further our formulation (4), we obtain

$$-i\, m\, \omega\, n\ \ v_1 = -e\, n_0\, E_1 - 3\, K\, T_e\, i\, k\, n_1. \tag{8}$$

Adding the equations (6), (7) (the equation of continuity and the Poisson equation)

$$-i\, \omega\, n_1 = -\, n_0\, i\, k\, v_1$$

$$i\, k\, \varepsilon_0\, E_1 = -e\, n_1$$

with the solution

$$n_1 = \frac{n_0\, i\, k\, v_1}{i\, \omega}\ ; \quad E_1 = -\frac{e}{i\, k\, \varepsilon_0} \cdot \frac{n_0\, i\, k\, v_1}{i\, \omega}$$

and inserting into (8), we obtain

$$i\, m\, \omega\, n_0\, v_1 = \left[e\, n_0 \left(-\frac{e}{i\, k\, \varepsilon_0} \right) + 3\, K\, T_e\, i\, k \right] \frac{n_0\, i\, k}{i\, \omega}\, v_1$$

with the solution

$$\omega^2\, v_1 = \left(\frac{n_0\, e^2}{\varepsilon_0\, m} + \frac{3\, K\, T_e}{m}\, k^2 \right) v_1 \quad \text{or}$$

$$\omega^2 = \omega_{pe}^{\ \ 2} + \frac{3\, K\, T_e}{m}\, k^2 \qquad \text{or}$$

$$\omega^2 = \omega_{pe}^{\ \ 2} + \frac{3}{2}\, k^2\, v_{th}$$

where $\frac{1}{2}\, m\, v_{th}^{\ \ 2} = K\, T_e$.

The condition for the wave propagation therefore reads:

$$\omega > \omega_{pe}.$$

Electromagnetic waves with $\vec{B}_0 = 0$.

Untill now we have discussed electrostatic waves (i.e., waves with zero magnetic component, $\vec{B}_1 = 0$.) Now we shall (at least partly), discuss waves with $\vec{B}_1 \neq 0$, electromagnetic waves, which propagates in a plasma without external magnetic field.

Let us start with simple light waves in a vacuum.

Since in vacuum are no currents $(\vec{j} = 0)$, and since $\varepsilon_0 \, \mu_0 = \frac{1}{c^2}$, the relevant Maxwell equations are

$$\nabla \times \vec{E}_1 = -\dot{\vec{B}}_1 \tag{1}$$

$$c^2 \, \nabla \times \vec{B}_1 = \dot{\vec{E}}_1 \tag{2}$$

(Here, $\dot{x} = \frac{\partial x}{\partial t}$).

Taking the curl of the Eq. (2) and substituting into the time derivative of Eq. (1), we obtain

$$c^2 \, \nabla \times \left(\nabla \times \vec{B}_1\right) = \nabla \times \dot{\vec{E}}_1 = -\dddot{\vec{B}}_1 \tag{3}$$

Expressing (using vector analysis),

$$\nabla \times \left(\nabla \times \vec{B}_1\right) = \nabla\left(\nabla \cdot \vec{B}_1\right) - \nabla^2 \vec{B}_1 \tag{4}$$

and considering the wave in the usual form $\vec{B}_1 \approx e^{i(k\,x - \omega\, t)}$, we obtain from (4)

$$\nabla \times \left(\nabla \times \vec{B}_1\right) = 0 - \frac{\partial^2}{\partial x^2} \vec{B}_1 = +k^2 \vec{B}_1 \tag{5}$$

Since $\dddot{\vec{B}}_1 = -\omega^2 \, \vec{B}_1$, the equation (3) gives

$$\omega^2 \, \vec{B}_1 = c^2 k^2 \vec{B}_1$$

and, therefore,

$$\omega^2 = k^2 \, c^2 . \tag{6}$$

Now, let us consider electromagnetic waves, propagating in plasma without external magnetic field.

The first Maxwell equation remain unchanged:

$$\nabla \times \vec{E}_1 = -\dot{\vec{B}}_1 \tag{7}$$

Nevertheless, the second Maxwell equation is changed, due to the current, generated by the particle motion

$$c^2 \nabla \times \vec{B}_1 = \frac{\vec{j}_1}{\varepsilon_0} + \dot{\vec{E}}_1 \tag{8}$$

The time derivative of this equation is

$$c^2 \nabla \times \dot{\vec{B}}_1 = \frac{\dot{\vec{j}}_1}{\varepsilon_0} \frac{\partial}{\partial t} \dot{\vec{j}}_1 + \ddot{\vec{E}}_1 . \tag{9}$$

Let us take the curl of Eq. (7)

$$\nabla \times \left(\nabla \times \vec{E}_1\right) = \nabla \left(\nabla \cdot \vec{E}_1\right) - \nabla^2 \vec{E}_1 = -\nabla \times \dot{\vec{B}}_1 . \tag{10}$$

Eliminating the term $\nabla \times \dot{\vec{B}}_1$ by means of Eq. (9) and using again the form $\approx e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)}$, we obtain

$$-\vec{k}\left(\vec{k} \cdot \vec{E}_1\right) + k^2 \vec{E}_1 = \frac{i\,\omega}{\varepsilon_0\, c^2} \vec{j}_1 + \frac{\omega^2}{c^2} \vec{E}_1 . \tag{11}$$

We consider transverse waves, i.e., waves with perpendicular direction of the oscillating electric field relative to the wave vector. Therefore, $\vec{k} \cdot \vec{E}_1 = 0$ and the equation (11) takes the form

$$\left(\omega^2 - c^2 k^2\right) \vec{E}_1 = -\frac{i\,\omega\, \vec{j}_1}{\varepsilon_0} . \tag{12}$$

We must now express the current $\vec{j}_1$. Since we consider waves with high frequency, the ions can be considered as fixed. Consequently, the current will be determioned only from the electron fluid equation (defining the current)

$$\vec{j}_1 = -n_0\, e\, \vec{v}_e . \tag{13}$$

Using the linear approximation of the fluid equation of motion for the electron component and neglecting the effect of the pressure, we have

$$m \frac{\partial \vec{v}_{e1}}{\partial t} = - e \vec{E}_1$$

and, consequently,

$$\vec{v}_{e1} = \frac{e \vec{E}_1}{i m \omega} . \tag{14}$$

Inserting into Eq. (12), we obtain

$$\left(\omega^2 - c^2 k^2\right) \vec{E}_1 = \frac{i \omega}{\varepsilon_0} n_0 e \frac{e \vec{E}_1}{i m \omega} = \frac{n_0 e^2}{\varepsilon_0 m} \vec{E}_1 = \omega^2{}_{pe} \vec{E}_1 .$$

From this, the dispersion equation reads

$$\omega^2 = \omega^2{}_{pe} + c^2 k^2 . \tag{15}$$

The dispersion equation exhibits a phenomenon called *cutoff.* From (15) follows

$$k^2 = \frac{1}{c^2} \left(\omega^2 - \omega^2{}_{pe}\right) .$$

Consequently, the wave propagates only for $\omega \rangle \omega_{pe}$.

Alfvén waves

Alfvén waves belongs to the class of hydromagnetic waves with **freqences in the** range 1 MHz, propagating in a plasma with external magnetic field B_0.

We shall consider the following geometry:

The Cartesian coordinate system x, y, z.
The external homogeneous magnetic field $\vec{B}_0 \equiv \vec{B}_z$.
The electric field $\vec{E}_1$ of the wave and the current $\vec{j}_1$ are both perpendicular to $\vec{B}_0$ and are parallel to the axis x.
The magnetic field of the wave, $\vec{B}_1$ and the fluid velocity $\vec{v}_1$ (the latter generated selfconsistently by the wave) are parallel to the axis y.

Summarizing, we consider the gollowing geometry:

$$\vec{k} \parallel \vec{B}_0, \qquad \vec{E}_1, \vec{j}_1 \perp \vec{B}_0, \qquad \vec{B}_{1 \perp} \vec{E}_1.$$

From two first Maxwell equations, we obtain

$$\nabla \times \nabla \times \vec{E}_1 = -\vec{k}\left(\vec{k}.\vec{E}_1\right) + k^2 \vec{E}_1 = \frac{\omega^2}{c_2}\vec{E}_1 + \frac{i\omega}{\varepsilon_0 c^2}\vec{j}_1.$$

Since $\vec{k} = k\,\vec{z}_0$, $\vec{E}_1 = E_1\,\vec{x}_0$, the nontrivial component of this equation is given by the x- component, namely,

$$\varepsilon_0\left(\omega^2 - c^2k^2\right)E_1 = -i\omega\, j_1 = -i\omega\, n_0\, e\left(v_{ix} - v_{ex}\right). \qquad (1)$$

The components v_{ix}, v_{ex} are given by the corresponding fluid motion equations:
Ions (M is the mass of ions):

$$\mathrm{M}\frac{\partial \vec{v}_{i1}}{\partial t} = e\left(\vec{E}_1 + \vec{v}_1 \times \vec{B}_0\right) \qquad T_i = 0,\ p_i = 0.$$

This gives in components:

$$-i\omega\, M\, v_{ix} = eE_1 + ev_{iy}B_0$$

$$-i\omega\, M\, v_{iy} = -e\, v_{ix}B_0$$

and finally

$$v_{ix} = \frac{ie}{M\omega}\left(1 - \frac{\omega_{ci}{}^2}{\omega_2}\right) E_1$$

$$v_{iy} = \frac{e}{M\omega}\frac{\omega_{ci}}{\omega}\left(1 - \frac{\omega_{ci}{}^2}{\omega^2}\right) E_1$$

Analogously for electrons: here, $M \to m,\ e \to = e,\ \omega_{ci} \to \omega_{ce}$. Moreover, we suppose $\omega_{ce}{}^2 >> \omega^2$. From that

$$v_{ex} = \frac{ie}{m\omega}\frac{\omega}{\omega_{ce}{}^2}{}^2 E_1 \doteq 0$$

$$v_{ey} = -\frac{e}{m}\frac{\omega_{ce}}{\omega^2}\frac{\omega^2}{\omega_{ce}{}^2} E_1 = -\frac{E_1}{B_0}$$

Inserting into (1), we obtain

$$\varepsilon_0\left(\omega^2 - c^2k^2\right)E_1 = -i\,\omega\,n_0\,e\,\frac{ie}{M\omega}\left(1 - \frac{\omega_{ci}{}^2}{\omega^2}\right)^{-1} E_1 .$$

Using the expression for ion plasma frequency ω_{pi} (analogously to electron plasma frequency)

$$\omega_{pi}{}^2 = \frac{n_0\,e^2}{\varepsilon_0 M}$$

we obtain the dispersion relation in the form

$$\omega^2 - c^2\,k^2 = \omega_{pi}{}^2\left(1 - \frac{\omega_{ci}{}^2}{\omega^2}\right)^{-1} .$$

Approximating $\omega^2 << \omega_{ci}{}^2$, the dispersion relation reeds

$$\omega^2 - c^2\,k^2 = \omega_{pi}{}^2\left(1 - \frac{\omega_{ci}{}^2}{\omega^2}\right)^{-1} .$$

Approximating $\omega^2 << \omega_{ci}^2$, we obtain

$$\omega^2 - c^2 k^2 = -\omega^2 \frac{\omega_{pi}^2}{\omega_{ci}^2} = -\frac{n_0 e^2}{\varepsilon_0 M} \frac{M^2}{e^2 B_0^2} \doteq -\omega^2 \frac{\rho}{\varepsilon_0 B_0^2}$$

where ρ is the mass density, defined as

$$\rho = n_i M + n_e m \doteq n_0 (M + m) \doteq n_0 M.$$

Then

$$\omega^2 \left(1 + \frac{\rho}{\varepsilon_0 B_0^2} \right) - c^2 k^2 = 0$$

and (introducing $\varepsilon_0 \mu_0 = \frac{1}{c^2}$)

$$\frac{\omega^2}{k^2} = \frac{c^2}{1 + \frac{\rho}{\varepsilon_0 B_0^2}} = \frac{c^2}{1 + \frac{\rho \mu_0}{B_0^2} c^2}.$$

For a laboratory plasma,

$$1 + \frac{\mu_0 \rho c^2}{B_0^2} >> 1.$$

Then

$$\frac{\omega^2}{k^2} \doteq \frac{c^2}{\frac{\rho \mu_0 c^2}{B_0^2}} = \frac{B_0^2}{\mu_0 \rho}$$

and
$$\frac{\omega}{k} = \frac{B_0}{\sqrt{\mu_0 \rho}} = v_A$$

where v_A is the Alfén velocity.

Discussion:

For the given geometry, the magnetic component of the wave, B_y, is directed in the y direction and forms a sinusoidal ripple on the constant magnetic field B_z. The electric component of the wave, E_x, is directed in the x direction. The electric component causes together with the constant magnetic field a drift with the velocity v_y

$$\vec{v}_y = \vec{E}_x \times \vec{B}_0 ,$$

the same for electrons and ions.

The lines of force of the magnetic field are also moving. The velocity of this motion (in the y direction) is $\frac{\omega}{k}\frac{B_y}{B_0}$. The component B_y follows from the Maxwell equation

$$\nabla \times \vec{E} = -\dot{\vec{B}}$$

which gives

$$E_x = \frac{\omega}{k} B_y ,$$

Inserting the component B_y into the expression for the velocity of the motion of field lines, , we obtain the same velocity as the fluid velocity.

Concequently, the fluid and the field lines oscillate together as if the particles (or the mass of the plasma) were coupled with field lines. This effect is known as "plasma is frozen to lines of force".