Equilibrium and stability

Hydromagnetic equilibrium

An equilibrium is a state in which all the forces are balanced, so that a time-independent solution is possible. The equilibrium is stable or unstable according to whether small perturbations of the equilibrium are damped or amplified.

Hydrodynamic Rayleigh–Taylor instability of a heavy fluid supported by a light one.

For the discussion of stability is necessary first to find an equilibrium and only after that to discuss its stability. The equilibrium problem is a nonlinear complicated problem. The discussion of stability consists in the discussion of the equation of motion for small deviation from the equilibrium state. These equations are linear and, therefore, solvable and easier to treat them.

Let us start with some general concepts of the plasma equilibrium. For that, we shall use the MHD equations of motion in its simplest form:

$$\rho \frac{\partial v}{\partial t} = \vec{j} \times \vec{B} - \nabla p$$

For a steady state, we have

$$\nabla p = \vec{j} \times \vec{B} \qquad (1)$$

Adding the Maxwell equation

$$c^2 \nabla \times \vec{B} = \vec{j} \big/ \varepsilon_0 \qquad (2)$$

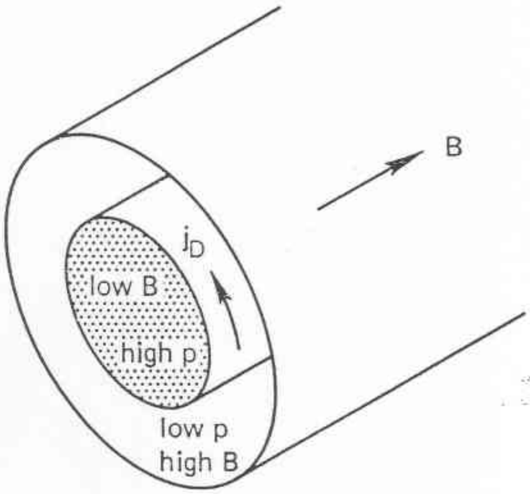
we shall be able to derive some important features.

A. The first equation (1) states that there is balance of forces between the pressure-gradient force and the Lorentz force. The following picture brings an explanation. The azimuthal current $\vec{j}$

(so called diamagnetic current) just balances the outward force of expansion. Its magnitude can be found by means of the vector multiplication by $\vec{B}$ of the first equation (the condition of the equilibrium)

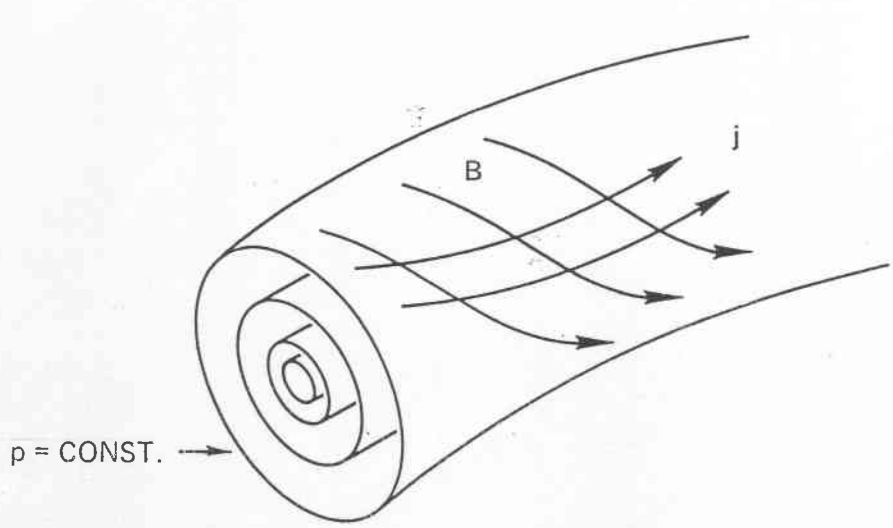
$$\vec{j}_{\perp} = \frac{\vec{B} \times \nabla p}{B^2} = \left(KT_i + KT_e\right)\frac{\vec{B} \times \nabla n}{B^2} \qquad (3)$$

From the MHD fluid view-point, the diamagnetic current is generated by the ∇p force across $\vec{B}$; the resulting current is just sufficient to balance the forces on each element of fluid and stop the motion.



In a finite-β plasma, the diamagnetic current significantly decreases the magnetic field, keeping the sum of the magnetic and particle pressures a constant.

B. Equation (1) tells that $\vec{j}$ and $\vec{B}$ are each perpendicular to ∇p. Since $\vec{j}$ and $\vec{B}$ are perpendicular to ∇p, they must lie on the surfaces of constant p.



Both the j and B vectors lie on constant-pressure surfaces.

C. Consider the component of Eq. (1) along the field line. It is

$$\frac{\partial p}{\partial s}=0$$

where s is the coordinate along the field line. For constant KT, this means that in hydromagnetic equilibrium the density is constant along a field-line of force.

Magnetic pressure

Let us substitute Eq. (2) into Eq. (1). We obtain

$$\nabla p=\varepsilon_0 c^2\left(\nabla\times\vec{B}\right)\times\vec{B}=\varepsilon_0 c^2\left[\left(\vec{B}.\nabla\right)\vec{B}-\frac{1}{2}\nabla B^2\right].$$

After some algebra

$$\nabla\left(p+\frac{1}{2}\varepsilon_0 c^2 B^2\right)=\varepsilon_0 c^2\left(\vec{B}.\nabla\right)\vec{B}.$$

In most cases, the right-hand size vanishes. Then

$$p+\frac{1}{2}\varepsilon_0 c^2 B^2=const.$$

The term $\frac{1}{2}\varepsilon_0 c^2 B^2$ is the magnetic field pressure.

The ratio β of the particle pressure and of the magnetic pressure is sometimes used

$$\beta=\frac{n\sum KT}{\frac{1}{2}\varepsilon_0 c^2 B^2}.$$

Usually, its magnitude is of the order $10^{-3}-10^{-1}$.

Instabilities

Classification:

1. Streaming instabilities – penetration of two fluids. The drift energy is used to excite waves, and the oscillation energy is gained at the expense of the drift energy in the unperturbed state.

2. Rayleigh-Taylor instability – in plasmas with a density gradient or with a sharp

boundary under the influence of an external nonelectromagnetic force (e.g., gravitational force). This force drives the instability. An example – instability of a heavy fluid supported by a light one.

3. Universal instabilities – plasma is not in perfect thermodynamic equilibrium as long as it is confined. The plasma pressure tends to make the plasma expand, and the expansion energy can drive an instability. This kind of free energy is always present in any finite plasmas.

4. Kinetic instabilities – if the distribution of particles is not Maxwellian, there is a deviation from thermodynamic equilibrium and instabilities can be driven by the anisotropy of the velocity distribution.

Examples:

1. TWO STREAM INSTABILITY

Let us consider a uniform plasma, in which ions are stationary and electrons have a velocity $\vec{v}_0$ relative to the ions. Let the plasma be cold, and without magnetic field. Then the linearized equations of motion are

$$Mn_0 \frac{\partial \vec{v}_{i1}}{\partial t} = en_0 \vec{E}_1$$

$$mn_0 \left[\frac{\partial \vec{v}_{e1}}{\partial t} + (\vec{v}_0 . \nabla) \vec{v}_{e1} \right] = -en_0 \vec{E}_1$$

The term $(\vec{v}_{e1} . \nabla) \vec{v}_0$ has been dropped because we assume $\vec{v}_0$ to be uniform. The term $(\vec{v}_0 . \nabla) \vec{v}_{i1}$ does not appear because we have taken $\vec{v}_{i0} = 0$.

Let us take the electrostatic wave in the form

$$\vec{E}_1 = Ee^{i(kx - \omega t)} \vec{x}_0$$

where $\vec{x}_0$ is the unit vector in the x=direction, parallel to $\vec{k}, \vec{v}_0$. The foregoing equations give

$$-i\omega Mn_0 \vec{v}_{i1} = en_0 \vec{E}_1 \qquad \vec{v}_{i1} = \frac{ie}{M} E \vec{x}_0$$

$$mn_0 (-i\omega + ikv_0) \vec{v}_{e1} = -en_0 \vec{E}_1 \qquad \vec{v}_{e1} = -\frac{ie}{m} \frac{E\vec{x}_0}{\omega - kv_0}.$$

Since the velocities $\vec{v}_{j1}$ are in the x-direction, we can omit the subscription x.

The equation of continuity for ion give

$$\frac{\partial n_{i1}}{\partial t} + n_0 \nabla . \vec{v}_{i1} = 0 \qquad\qquad n_{i1} = \frac{k}{\omega} n_0 v_{i1} = \frac{ien_0 k}{M\omega^2} E .$$

The other terms in $\nabla .(n\vec{v}_i)$ vanish because $\nabla n_0 = \vec{v}_{0i} = 0$. The electron continuity equation is

$$\frac{\partial n_{e1}}{\partial t} + n_0 \nabla . \vec{v}_{e1} + (\vec{v}_0 . \nabla) n_{e1} = 0$$

$$(-i\omega + ikv_0) n_{e1} + ikn_0 v_{e1} = 0$$

$$n_{e1} = \frac{kn_0}{\omega - kv_0} v_{e1} = -\frac{iekn_0}{m(\omega - kv_0)} E .$$

Since the unstable waves are high-frequency oscillations, we may not usethe plasma approximation, but we must use Poisson's equation

$$\nabla . \vec{E}_1 = \frac{e}{\varepsilon_0} (n_{i1} - n_{e1}) .$$

Then

$$ikE = \frac{e}{\varepsilon_0} (ien_0 kE) \left[\frac{1}{M\omega^2} + \frac{1}{m(\omega - kv_0)^2} \right] .$$

This then gives the dispersion relation

$$1 = \omega_p{}^2 \left[\frac{m/M}{\omega^2} + \frac{1}{(\omega - kv_0)^2} \right] .$$

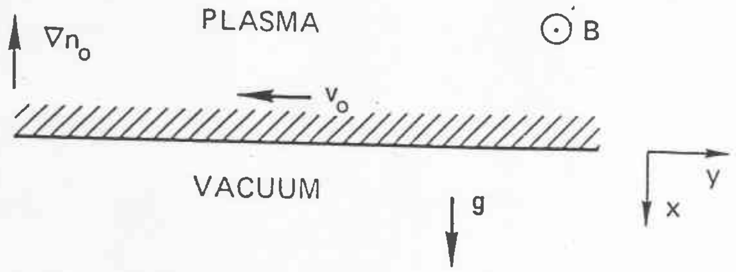
This is the algebraic equation of the fourth order. After some complicated algebra, we obtain the increment of the order

$$\mathrm{Im}\left(\frac{\omega}{\omega_p} \right) \approx \left(\frac{m}{M} \right)^{1/3} .$$

THE "GRAVITATIONAL", "RAYLEIGH-TAYLOR" INSTABILITY

In a plasma, a Rayleigh-Taylor instability can occur because the magnetic field acts as a light fluid supporting a heavy fluid (the plasma). In curved magnetic fields, the centrifugal force on the plasma due to particle motion along the curved lines of force acts as an equivalent "gravitational" force.

To treat the simplest case, consider a plasma boundary lying in the *y-z* plane.



A plasma surface subject to a gravitational instability.

Let there be a density gradient ∇n_0 in the –x direction and a gravitational field $\bar{g}$ in the –x direction. Let $KT_i = KT_e = 0$ and let the magnetic field $\bar{B}_0$ (in the –z direction) is constant. In the equilibrium state, the ions obey the equation (obtained form the fluid equation of motion)

$$Mn_0(\bar{v}_0.\nabla)\bar{v}_0 = en_0\bar{v}_0 \times \bar{B}_0 + Mn_0\bar{g}. \qquad (1)$$

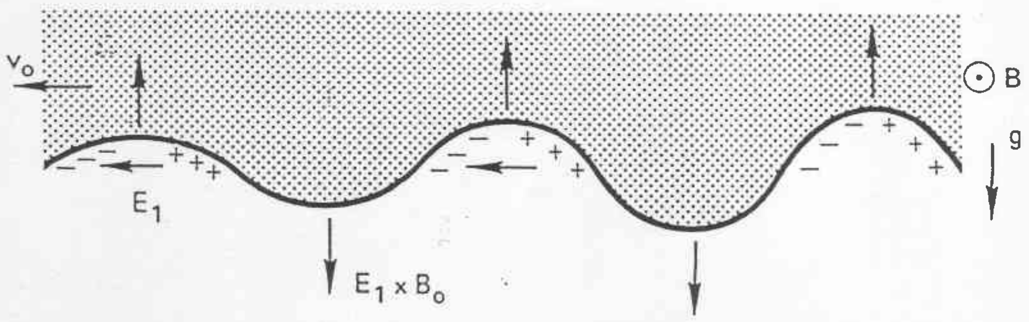
For $\bar{v}_0$ constant, and taking the cross-product of the foregoing equation with $\bar{B}_0$, we obtain

$$\bar{v}_0 = \frac{M}{e}\frac{\bar{g}\times\bar{B}_0}{B_0{}^2} = -\frac{g}{\Omega_c}\bar{y}_0 . \qquad (2)$$

Here, Ω_c is the ion cyclotron frequency.

The drift of electrons can be neglected due to the limit $m/M \to 0$.

Let us assume that the boundary of the plasma is perturbed by a ripple as seen in the following picture.



Physical mechanism of the gravitational instability.

The drift $\bar{v}_0$ will cause the ripple to grow. The drift of ions causes a charge to build up on the sides of the ripple and an electric field develops which changes sign along the axis y. The ripple grows as a result of these properly phased $\bar{E}\times\bar{B}$ drifts.

To find the growth rate, we shall use the usual linearized analysis for waves propagating in the y direction, $\bar{k} = k\bar{y}_0$. The perturbed equation of ion's motion reeds:

$$M(n_0+n_1)\left[\frac{\partial}{\partial t}(\bar{v}_0+\bar{v}_1)+(\bar{v}_0+\bar{v}_1).\nabla(\bar{v}_0+\bar{v}_1)\right]=e(n_0+n_1)\left[\bar{E}_1+(\bar{v}_0+\bar{v}_1)\times\bar{B}_1\right]+$$
$$+M(n_0+n_1)\bar{g} \tag{3}$$

Multiplying (1) by $1+{n_1}/{n_0}$, we obtain

$$M(n_0+n_1)(\bar{v}_0.\nabla)\bar{v}_0 = e(n_0+n_1)\bar{v}_0\times\bar{B}_0 + M(n_0+n_1)\bar{g}\,. \tag{4}$$

Let us subtract the foregoing equation from Eq. (3) and neglect second-order terms.
We then obtain

$$Mn_0\left[\frac{\partial\bar{v}_1}{\partial t}+(\bar{v}_0.\nabla)\bar{v}_1\right]=en_0\left(\bar{E}_1+\bar{v}_1\times\bar{B}_0\right).$$

For perturbation of the form $\exp[i(ky-\omega t)]$ we have

$$M(\omega-kv_0)\bar{v}_1 = ie\left(\bar{E}_1\right)+\bar{v}_1\times\bar{B}_0$$

For $E_x = 0$ and for

$$\Omega_c^{\,2} >> (\omega-kv_0)^2$$

the solution is

$$v_{ix}=\frac{E_y}{B_0};\quad v_{iy}=-i\frac{\omega-kv_0}{\Omega_c}\frac{E_y}{B_0} \tag{5}$$

The corresponding quantity for electrons vanishes in the limit ${m}/{M}\to 0$. For the electrons, we therefore have

$$v_{ex}={E_y}/{B_0};\quad v_{ey}=0. \tag{6}$$

The perturbed equation of continuity for ions offers:

$$\frac{\partial n_1}{\partial t}+\nabla.(n_0v_0)+(\bar{v}_0.\nabla)n_1+n_1\nabla.\bar{v}_0+(\bar{v}_1.\nabla)n_0+n_0\nabla.\bar{v}_1+\nabla.(n_1\bar{v}_1)=0$$

The zeroth-order term vanishes since $\vec{v}_0$ is perpendicular to ∇n_0 and the term $n_1 \nabla . \vec{v}_0$ vanishes if $\vec{v}_0 = const$. The first-order equation for ions is, therefore,

$$-i\,\omega\, n_1 + i\,k\,v_0\,n_1 + v_{ix}\,n_0^{'} + i\,k\,n_0\,v_{iy} = 0 . \qquad (7)$$

The electrons follow a simpler equation, since $\vec{v}_{e0} = 0$ and $v_{ey} = 0$. Then

$$-i\omega\,n_1 + v_{ex}\,n_0^{'} = 0 \qquad (8)$$

We use there the plasma approximation and we have assumed that $n_{i1} = n_{e1}$. This is possible because the unstable waves are of low frequency.

Equations (5) and (7) yield

$$(\omega - k\,v_0)\,n_1 + i\frac{E_y}{B_0}n_0^{'} + i\,k\,n_0\frac{\omega - k\,v_0}{\Omega_c}\frac{E_y}{B_0} = 0 . \qquad (8)$$

Equations (6) and (8) yield

$$\omega\,n_1 + i\frac{E_y}{B_0}n_0^{'} = 0 \qquad\qquad \frac{E_y}{B_0} = \frac{i\omega n_1}{n_0^{'}} .$$

Lert us substitute this into Eq. (8). We obtain after some algebra

$$\omega\,(\omega - kv_0) = -v_0\Omega_c n_0^{'}\frac{1}{n_0} .$$

Substituting there for v_0 from Eq. (2), we obtain a quadratic equation for ω

$$\omega^2 - k\,v_0\,\omega - g\left(n_0^{'}/n_0\right) - 0.$$

This gives

$$\omega = \frac{1}{2}kv_0 +/- \left[\frac{1}{4}k^2 v_0^{\,2} + g\left(n_0^{'}/n_0\right)\right]^{1/2} .$$

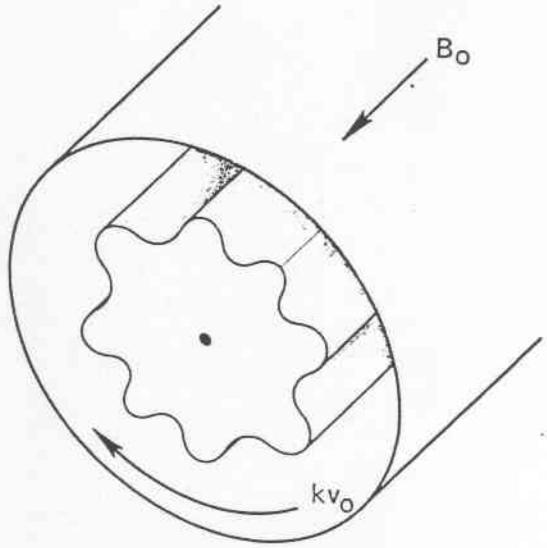
The instability requires

$$-g\,{}^{n_0^{'}}\!/_{n_0} > \frac{1}{4}k^2 v_0^{\,2}$$

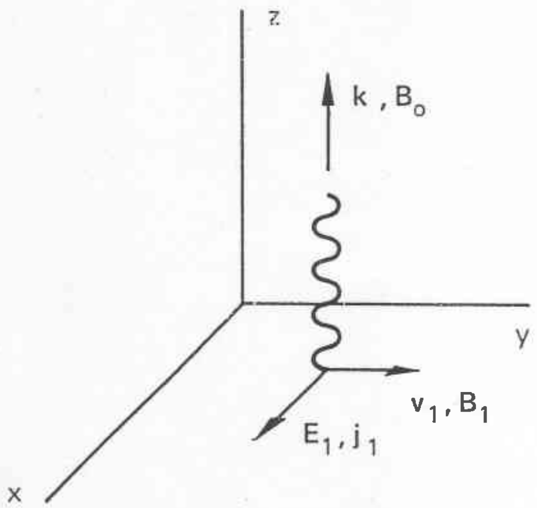
with the growth rate

$$\gamma = \mathrm{Im}(\omega) = \left[-g\left({}^{n_0^{'}}\!/_{n_0}\right)\right]^{1/2} .$$

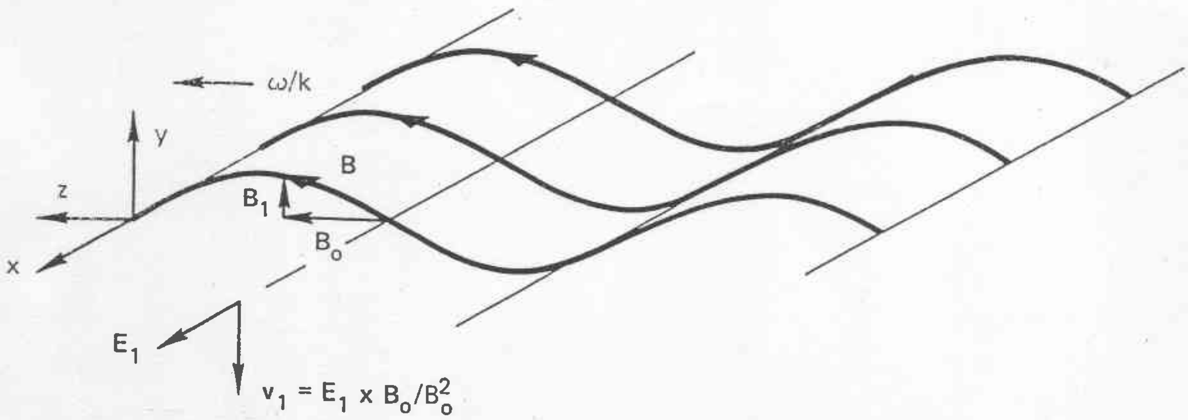
This instability is sometimes called a "flute" instability . The surface of constant density resembles fluted Greek column – see the picture



A "flute" instability.



Geometry of an Alfvén wave propagating along B_0.



Relation among the oscillating quantities in an Alfvén wave and the (exaggerated) distortion of the lines of force.