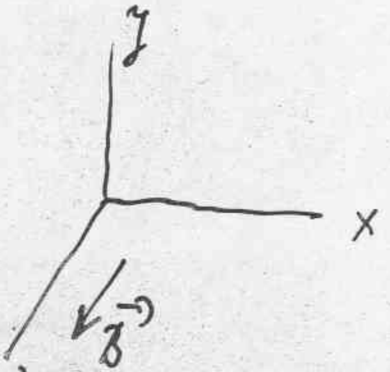
Single particle motion

1. Motion of a charged particle in an uniform magnetic field $\vec{B}$; $\vec{E} = 0$.



$$\vec{B} = B\,\vec{z}_0$$

$$m \frac{d\vec{v}}{dt} = q\,(\vec{v} \times \vec{B})$$

$$m\dot{v}_x = qBv_y \; ; \; m\dot{v}_y = -qBv_x \; ; \; m\dot{v}_z = 0$$

$$\left(\dot{r} = \frac{dr}{dt} \right)$$

Differentiating

$$\ddot{v}_x = +\frac{q}{m} B \dot{v}_y \quad ; \quad \ddot{v}_y = -qB\dot{v}_x$$

$$\Rightarrow \quad \ddot{v}_x = \frac{q}{m} B \left(-\frac{q}{m} B \right) v_x \Rightarrow \ddot{v}_x = -\left(\frac{q}{m} B \right)^2 v_x$$

$$\ddot{v}_y = -\frac{q}{m} B \left(\frac{q}{m} B v_y \right) v_y \Rightarrow \ddot{v}_y = -\left(\frac{q}{m} B \right)^2 v_y$$

$$\omega_c = \frac{|q| B}{m}$$

$$v_{x,y} = v_\perp \exp\left(\pm i \omega_c t + i \delta_{x,y} \right)$$

let choose

$$v_x = v_\perp e^{i\omega_c t}$$

$$v_y = \frac{m}{qB} \dot{v}_x = \frac{m}{qB} v_\perp i \omega_c e^{i\omega_c t} = \pm v_\perp i e^{i\omega_c t}$$

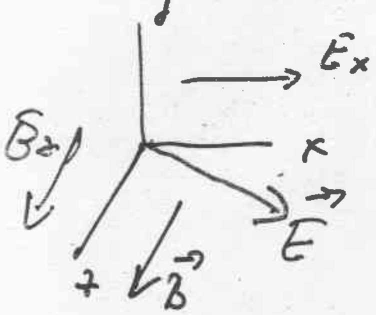
$$\Rightarrow \quad y - y_0 = \pm \frac{v_\perp}{\omega_c} e^{i\omega_c t}$$

$$x - x_0 = -i \frac{v_\perp}{\omega_c} e^{i\omega_c t}$$

$$r_L = \frac{v_\perp}{\omega_c}$$ – Larmor radius

$$m \dot{v}_z = 0 \Rightarrow z = z_0 + v t \quad ; \quad v_\perp, v \text{ constants}$$

2. Motion of a charged particle in an uniform magnetic field $\vec{B}$ and in an uniform electric field $\vec{E}$



$$m \frac{d\vec{v}}{dt} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

Due our choice

$$\frac{dv_z}{dt} = \frac{q}{m} E_z \Rightarrow z = \frac{q}{2m} E_z t^2 + v_{z0} t$$

acceleration

$$\frac{dv_x}{dt} = \frac{q}{m} E_x \pm \omega_c v_y \; ; \quad \frac{dv_y}{dt} = \mp \omega_c v_x$$

Differentiating

$$\ddot{v}_x = -\omega_c^2 v_x$$

$$\ddot{v}_y = \mp \omega_c \left(\frac{q}{m} E_x \pm \omega_c v_y \right) = -\omega_c^2 \left(\frac{E_x}{B} + v_y \right).$$

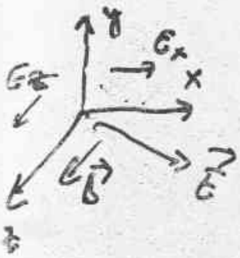
$$\frac{d^2}{dt^2}\left(v_y + \frac{E_x}{B} \right) = -\omega_c^2 \left(v_y + \frac{E_x}{B} \right)$$

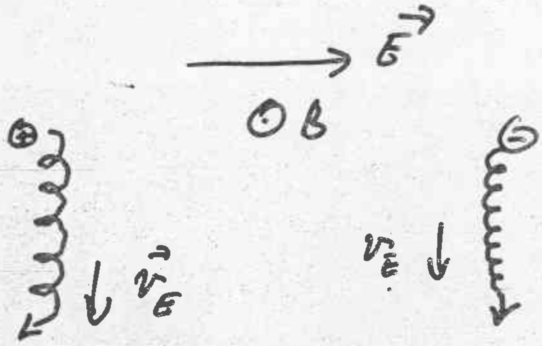
$$\Rightarrow \quad v_x = v_\perp e^{i\omega_c t}$$

$$v_y = \pm i v_\perp e^{i\omega_c t} - \frac{E_x}{B} \; ;$$

v_E - drift of the guiding center, $v_E = \dfrac{E\,(\mathrm{V m^{-1}})}{B\,(\mathrm{tesla})} \ \dfrac{m}{s}$

$$\vec{v}_E = \frac{\vec{E} \times \vec{B}}{B^2}$$





v_E independent of q, m, $v_\perp$.

$$\frac{m}{q}\dot{\vec{v}} = \left(\vec{E} + \vec{v}\times\vec{B}\right)$$

$$\frac{m}{q}\dot{\vec{v}}_\perp = \left(\vec{E}_\perp + \vec{v}_\perp\times\vec{B}\right)$$

$$\vec{v}_\perp = \vec{c}_\perp + \vec{V}$$

$$\frac{m}{q}\dot{\vec{c}}_\perp = \vec{E}_\perp + \vec{V}\times\vec{B} + \vec{c}_\perp\times\vec{B}$$

Let us choose

$$\vec{V} = \frac{\vec{E}\times\vec{B}}{B^2}$$

Then

$$\frac{m}{q}\dot{c}_\perp = \vec{c}_\perp\times\vec{B} \longrightarrow \text{uniform rotation}$$

$$\vec{E}_\perp + \vec{V}\times\vec{B} = 0 \Rightarrow \vec{E}_\perp\times\vec{B} + (\vec{V}\times\vec{B})\times\vec{B} =$$

$$= \vec{E}\times\vec{B} - \vec{B}\times(\vec{V}\times\vec{B}) = \vec{E}\times\vec{B} - \vec{V}(\vec{B}\cdot\vec{B}) + \vec{B}(\vec{B}\cdot\vec{V}) = 0$$

$$\vec{V} = \vec{V}_\perp \Rightarrow \vec{E}\times\vec{B} - \vec{V}_\perp B^2 = 0 \Rightarrow \vec{V}_\perp = \vec{V} = \frac{\vec{E}\times\vec{B}}{B^2}$$

Motion of a charged particle in a field of a general force $\vec{F}$

$\vec{E}, \vec{B}$ $\qquad \vec{v}_E = \dfrac{\vec{E} \times \vec{B}}{B^2}$

$$q\vec{E} \rightarrow \vec{F}$$

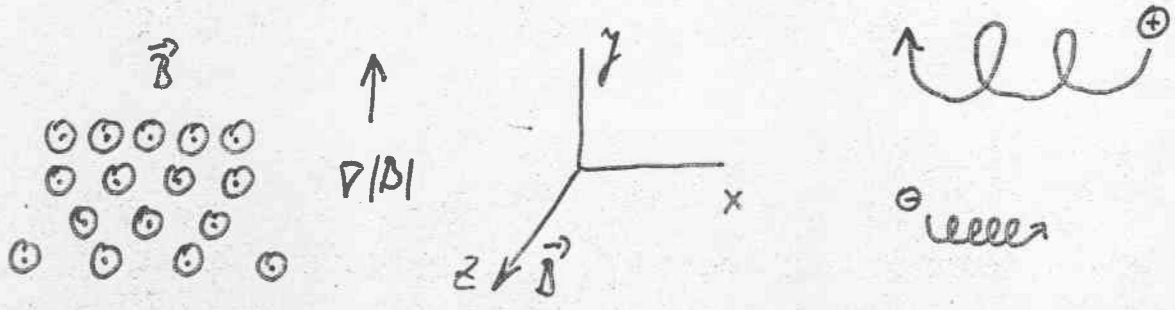
$$\Rightarrow \quad \vec{v}_F = \frac{1}{q} \frac{\vec{F} \times \vec{B}}{B^2}$$

e.g. $\vec{F} = m\vec{g} \quad \Rightarrow \quad \vec{v}_g = \dfrac{m}{q} \dfrac{\vec{g} \times \vec{B}}{B^2}$.

Drift velocity $\vec{v}_F$ depends on the charge q.

Motion of a charged particle in a nonuniform magnetic field $\vec{B}$; $\vec{E} \equiv 0$.

a) grad $\vec{B}$ drift



$$r_L = \frac{v_\perp}{\omega_c}$$

$$(\vec{u}\cdot\nabla)\vec{G} = u_x\frac{\partial G}{\partial x} + u_y\frac{\partial G}{\partial y} + u_z\frac{\partial G}{\partial z}$$

$$\vec{F} = q\,\vec{v}\times\vec{B} \quad ; \quad \frac{\partial B_z}{\partial y} \neq 0$$

Expanding $\vec{B}$ into a Taylor series

$$\vec{B} = \vec{B}_0 + (\vec{r}\cdot\nabla)\vec{B} \ldots$$ (in general), scalar diff. operator

$$B_z = B_0 + \frac{\partial B_z}{\partial y}\, y + \cdots$$ (for our approximation)

we obtain the y- component of the Lorenz force

$$F_y = -q\,v_x\,B_z(y) = -q\,v_\perp \cos\omega_c t\,\left[B_0 \pm r_L \cos\omega_c t \cdot \frac{\partial B_z}{\partial y}\right]$$

We have introduced for v_x

$$v_x = v_\perp \cos\omega_c t$$

and the expression for the y- coordinate

$$y = \pm\frac{v_\perp}{\omega_c}\cos\omega_c t = \pm r_L \cos\omega_c t\,.$$

The expansion, of course, has its sence only for $\frac{r_L}{L} \ll 1$

(L is the scale length of the inhomogenity).

We calculate the average values of the foregoing expression; averaging in τ_c

$$\tau_c = \frac{2\pi}{\omega_c}\,.$$

After averaging

$$\overline{F_y} = \mp q\,v_\perp\,r_L\,\frac{1}{2}\,\frac{\partial B_0}{\partial y}$$

$$\overline{F_x} = 0 \quad ; \quad F_z \equiv 0$$

Introducing the component F_y into the general expression for the drift in the magnetic field and under the force $\vec{F}$

$$\vec{v}_F = \frac{1}{q}\,\frac{\vec{F}\times\vec{B}}{B^2}$$

we obtain for our special choice

$$v_F = \frac{1}{q}\,\frac{F_y}{B}\,\vec{x}_0 = \mp\,\frac{v_\perp r_L}{B}\,\frac{1}{2}\,\frac{\partial B}{\partial y}\,\vec{x}_0$$

$\vec{x}_0$ is the unit vector in x-direction.

More generally, the drift due to the nonuniformity of the magnetic field

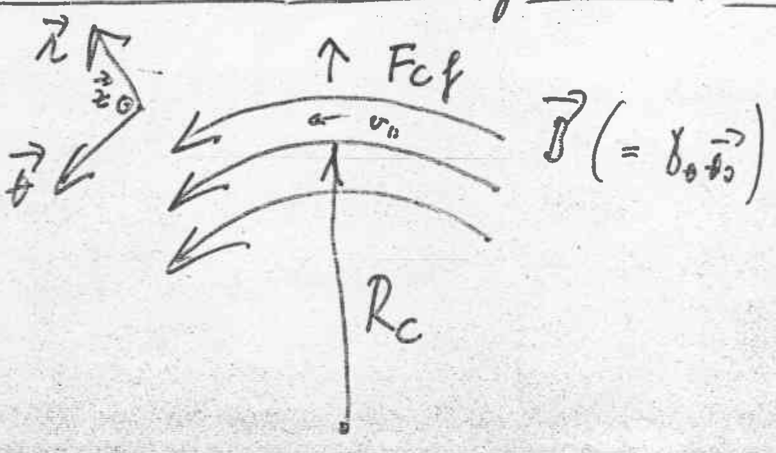
$$\boxed{\vec{v}_{\nabla B} = \pm\,\frac{1}{2}\,v_\perp r_L\,\frac{\vec{B}\times\nabla B}{B^2}}$$

Drift of ions is in the opposite direction than of electrons

This drift is called <u>grad B-drift</u>.

b) <u>The drift due to the curved magnetic field $\vec{B}$</u>

The geometry



$v_\parallel$ ... the average parallel component of the velocity $\vec{v}$ along $\vec{B}$

R_c .. the radius of curvature of the field line

The centrifugal force $\vec{F}_{cf}$, acting on the particle, is

$$\vec{F}_{cf} = \frac{m v_\parallel^2}{R_c} \vec{r}_0 \qquad (\vec{r}_0 \text{ unit radial vector})$$

$$v_r = \frac{1}{q} \frac{\vec{F}_{cf} \times \vec{B}}{B^2} = \frac{m v_\parallel^2}{q B^2} \frac{\vec{R}_c \times \vec{B}}{R_c^2} .$$

$\vec{v}_r$ ... the curvature drift.

<u>But</u> – there must be also a gradient of the field $\Rightarrow$

$\Rightarrow$ also the grad B drift

Let us estimate the gradient of $\vec{B}$

In a vacuum, the Maxwell equations require

$$\nabla \times \vec{B} = 0$$

In our cylindrical system, $\nabla \times \vec{B}$ has only z-component

$$(\nabla \times \vec{B})_z = \frac{1}{r} \frac{\partial}{\partial r} (r B_\theta) = 0$$

$$\Rightarrow \quad r B_\theta = \text{const} \quad \Rightarrow \quad B_\theta = \frac{\text{const}}{r} .$$

$$\left[\nabla \times \vec{B} = \left(\frac{1}{r} \frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial z} \right) \vec{r}_0 + \left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} \right) \vec{\theta}_0 + \left(\frac{1}{r} \frac{\partial (r B_\theta)}{\partial r} - \frac{1}{r} \frac{\partial B_r}{\partial \theta} \right) \vec{z}_0 \right.$$

Consequently, $|B| = \frac{\text{const}}{R_c}$; $\nabla |B| = - \frac{\text{const}}{R_c^2} \frac{\vec{R}_c}{R_c}$

$$\frac{\nabla |B|}{|B|} = - \frac{\vec{R}_c}{R_c^2}$$

Inserting into the expression for the grad-B drift

$$\vec{v}_{\nabla B} = \pm \frac{1}{2} v_\perp r_L \frac{\vec{B} \times \nabla B}{B^2}$$

we obtain

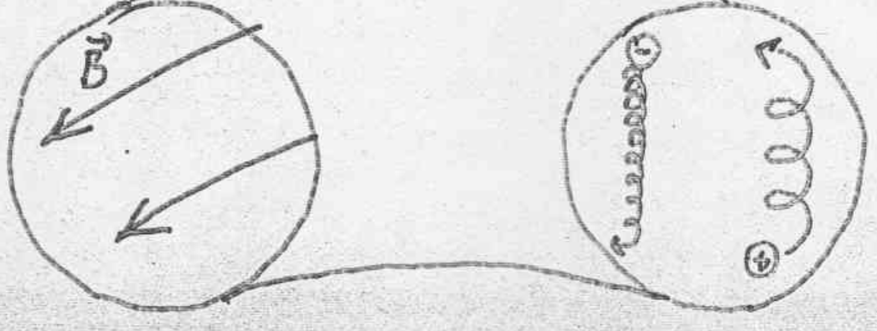
$$\vec{v}_{\nabla B} = \mp \frac{1}{2} \frac{v_\perp r_L}{B^2} \vec{B} \times |B| \frac{\vec{R}_c}{R_c^2} = \pm \frac{1}{2} \frac{v_\perp^2}{\omega_c} \frac{\vec{R}_c \times \vec{B}}{R_c^2 B}$$

($r_L = \frac{v_\perp}{\omega_c}$). Using for $\omega_c = \pm \frac{qB}{m}$, we obtain finally

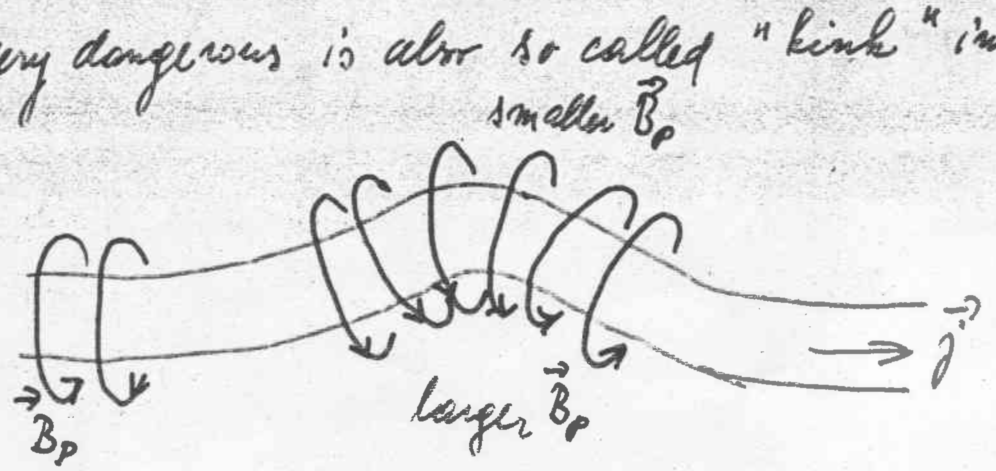
$$\vec{v}_{\nabla B} = \frac{1}{2} \frac{m}{q} v_\perp^2 \frac{\vec{R}_c \times \vec{B}}{R_c^2 B^2}$$

The <u>total</u> drift in a curved magnetic field is

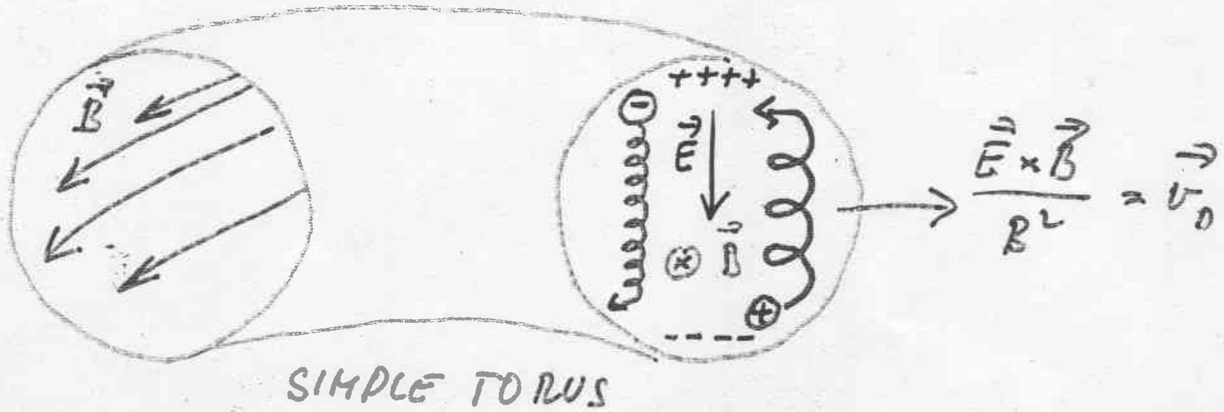
$$\boxed{\vec{v}_C = \vec{v}_R + \vec{v}_{\nabla B} = \frac{m}{q} \frac{\vec{R}_c \times \vec{B}}{R_c^2 B^2} \left(v_\parallel^2 + \frac{1}{2} v_\perp^2 \right)}$$



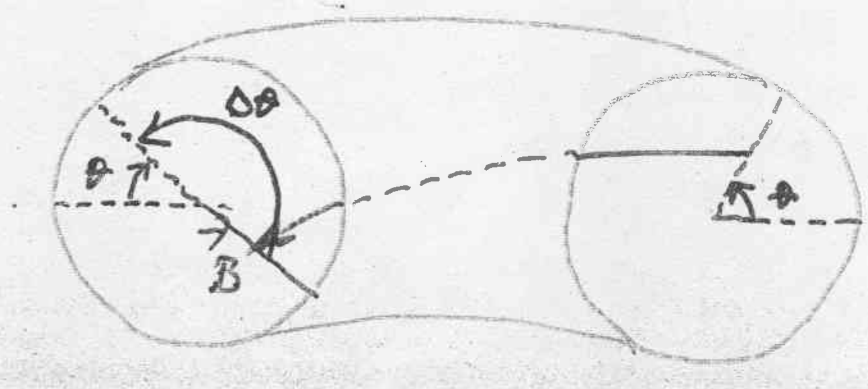
Very dangerous is also so called "kink" instability



The current flowing through the plasma generates the poloidal field $\vec{B}_p$. If a perturbation of this kind appears, the magnetic field $\vec{B}_p$ inside is larger than outside. The corresponding magnetic pressure then push the plasma towards the wall.



SIMPLE TORUS



Summary of drifts

$\vec{E}, \vec{B}$ $\quad \vec{v}_E = \dfrac{\vec{E} \times \vec{B}}{B^2}$

drift in a general force $\vec{F}$, $\vec{B}$ homogeneous $\quad \vec{v}_F = \dfrac{1}{q} \dfrac{\vec{F} \times \vec{B}}{B^2}$

grad B drift $\quad \vec{v}_{\nabla B} = \pm \dfrac{1}{2} v_\perp r_L \dfrac{\vec{B} \times \nabla B}{B^2}$

drift in magnetic field with curved field lines $\quad \vec{v}_R = \dfrac{m v_\parallel^2}{q} \dfrac{\vec{R}_k \times \vec{B}}{R_k^2 B^2}$

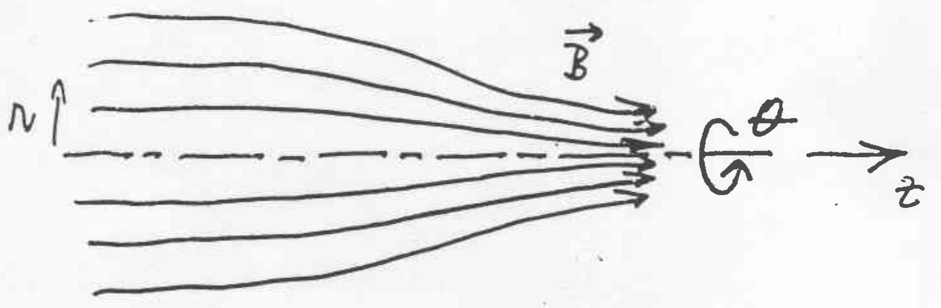
drift in an inhomogeneous magnetic field with curved field lines

$$\vec{v}_R + \vec{v}_{\nabla B} =$$

$$= \frac{m}{q} \left(v_\parallel^2 + \frac{1}{2} v_\perp^2 \right) \times$$

$$\times \frac{\vec{R}_k \times \vec{B}}{R_k^2 B^2}$$

Effect of magnetic mirrors on the particle motion



$$\frac{\partial}{\partial \theta} = 0$$

$$B_\theta = 0$$

$B_r \neq 0$ follows from $\nabla \cdot \vec{B} = 0$

In cylindrical coordinates this gives

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r B_r\right) + \frac{\partial B_z}{\partial z} = 0$$

$$\Rightarrow \quad r B_r = -\int_0^r r \frac{\partial B_z}{\partial z}\, dr \simeq -\frac{1}{2} r^2 \frac{\partial B_z}{\partial z}\Big|_{r=0}$$

$$\Rightarrow \quad B_r = -\frac{1}{2} r \frac{\partial B_z}{\partial z}\Big|_{r=0}$$

$$m \frac{d\vec{v}}{dt} = \vec{F} = q\, \vec{v} \times \vec{B}$$

$$F_r = q\left(v_\theta^{*} B_z - v_z \cancel{B_\theta}\right) \qquad \text{cyl. rotation}^{*}$$

$$F_\theta = q\left(-v_r^{*} B_z + v_z B_r\right) \qquad B_r(r=0) = 0$$

$$F_z = q\left(v_r \cancel{B_\theta} - v_\theta B_r\right)$$

Due to the axisymetry, $B_\theta = 0$.

$$F_z = -q\, v_\theta B_r$$

$$F_z = \frac{1}{2} q\, v_\theta\, r \frac{\partial B_z}{\partial z}$$

$$v_\theta = \mp v_\perp \quad ; \quad r_L = \frac{v_\perp}{\omega_c} \quad ; \quad \omega_c = \frac{|q| B}{m}$$

$$F_z = \mp \frac{1}{2} q\, v_\perp r_L \frac{\partial B_z}{\partial z} = \mp \frac{1}{2} q \frac{v_\perp^2}{\omega_c} \frac{\partial B_z}{\partial z} =$$

$$= \boxed{-\frac{1}{2} \frac{m v_\perp^2}{B} \frac{\partial B_z}{\partial z}}\,.$$

$$r_\perp = \frac{v_\perp}{\omega_c} \; ; \quad \omega_c = \frac{|q| B}{m}$$

$$\mu = \frac{1}{2} m v_\perp^2 \frac{1}{B}$$ .... the magnetic moment

$$F_z = -\mu \frac{\partial B_z}{\partial z}$$, or generally

$$F_\parallel = -\mu \frac{\partial B}{\partial s} = -\mu \nabla_\parallel B$$

where ds is a line element along $\vec{B}$.

$$m \frac{dv_\parallel}{dt} = -\mu \frac{\partial B}{\partial s}$$

$$v_\parallel = \frac{ds}{dt}$$; multiplying by $v_\parallel$, $\frac{ds}{dt}$

$$m v_\parallel \frac{dv_\parallel}{dt} = -\mu \frac{\partial B}{\partial s} \frac{ds}{dt} = -\mu \frac{dB}{dt}$$

$$m v_{\parallel} \frac{dv_{\parallel}}{dt} = \frac{d}{dt}\left(\frac{1}{2} m v_{\parallel}^2\right)$$

(*)

$$-\mu \frac{\partial B}{\partial s}\frac{ds}{dt} = -\mu \frac{dB}{dt} \Rightarrow \underline{\frac{d}{dt}\left(\frac{1}{2} m v_{\parallel}^2\right) = -\mu \frac{dB}{dt}}$$

The complete energy of the particle must be conserved ; this means

$$\frac{d}{dt}\left(\frac{1}{2} m v_{\parallel}^2 + \frac{1}{2} m v_{\perp}^2\right) = 0$$

$$\frac{1}{2} m v_{\perp}^2 = \mu B$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2} m v_{\parallel}^2 + \mu B\right) = 0$$

$$\underline{\frac{d}{dt}\left(\frac{1}{2} m v_{\parallel}^2\right)} + \frac{d}{dt}(\mu B) = 0$$

Inserting for the first term from (*)

$$\frac{d}{dt}\left(\frac{1}{2} m v_{\parallel}^{2}\right) = -\mu \frac{dB}{dt}$$

we obtain

$$-\mu \frac{dB}{dt} + \frac{d}{dt}(\mu B) = 0$$

This can be satisfied only for

$$\frac{d\mu}{dt} = 0$$

μ – adiabatic invariant

Kruskal M.D., : "Advanced Theory of Gyrating Particles", IAEA, Vienna 1965

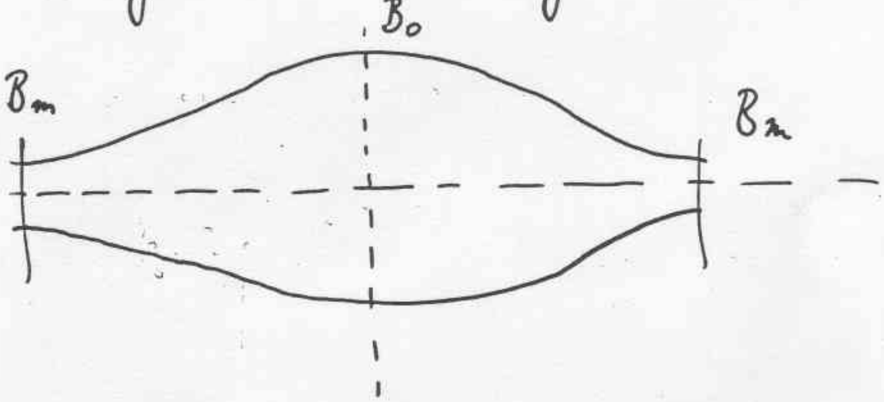
$$\frac{\Delta \omega_c}{\omega_c} < 1.$$

Ehrenfest

Violation under

1) fast changes of the magnetic field
2) under the influence of the radiofrequency waves with frequency $\omega = \omega_c$.

Trapping between magnetic mirrors



$$\mu = \frac{\frac{1}{2} m v_\perp^2}{B} = \text{const}$$

Not all particles are reflected !

The reflection requires sufficient $\frac{1}{2} m v_\perp^2$.