# BASIC FEATURES OF THE KINETIC THEORY OF PLASMA WAVES AND OF WAVES-PARTICLES INTERACTION

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**Abstract**

In the following lectures, a brief introduction into the kinetic theory of plasma waves and into the interaction of waves with plasma particles is presented.

## 1. THE BOLTZMANN KINETIC EQUATION

In this chapter, we shall derive the Boltzmann equation, the basic equation of plasma physics. We shall use the approach, commonly called the "BBGKY theory" (after Bogolyubov, Born, Green, Kirkwood and Yvon). As already mentioned [1], this approach is only one among the various theories of the nonequilibrium statistical mechanics, but it is a systematic and perhaps the most powerful theory.

Let us first summarize the assumptions under which this theory is commonly presented. Let a plasma be formed by $N$ electrons and $N$ singly-charged ions, and let us consider the ion component to be the uniform background of immobile ions. Let us assume that the particle interaction can be derived from the potential energy $\phi$ of the interaction, which for the mutual interaction of two charged particles is given as

$$\phi = \sum_{i<j=1} \phi_{ij}; \qquad \phi_{ij} = \frac{1}{4\pi\epsilon_0} \frac{e^2}{|\mathbf{q}_i - \mathbf{q}_j|} \tag{1}$$

where $\mathbf{q_i}$, $\mathbf{q_j}$ are the space vectors of both Coulomb interacting particles, labelled as $i$ and $j$; $e$ is the charge of the electron and $\epsilon_0$ is the permitivity of free space.

Let us define the exact $N$-particle distribution function od electrons $f_N^{exact}$ as [2]

$$f_N^{exact} = f_N^{exact}(\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_N, \mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_N, t), \tag{2}$$

where $\mathbf{q}_i$, $\mathbf{p}_i$ are the generalized coordinates and momenta, respectively, and where $t$ stands for time. The exact distribution function reads

$$f_N^{exact}(\mathbf{q}_i, \mathbf{p}_i, t) = \prod_{i=1}^{N} \delta(\mathbf{q}_i - \mathbf{q}_i(t))\, \delta(\mathbf{p}_i - \mathbf{p}_i(t)) \tag{3}$$

where $\delta(x)$ is the $\delta$-function, and where $\mathbf{q}(t)_i$, $\mathbf{p}(t)_i$ represent the phase trajectory.

Although $f_N^{exact}$ represents formally the exact distribution, its practical importance is less significant for the following reason. The determination of $f_N^{exact}$ requires the knowledge of the dynamics of all $N$ particles. The lack of the knowledge of all initial conditions makes this determination impossible. Therefore, a statistical representation, based on a probability approach, is more suitable. We shall, therefore, replace the exact distribution function $f_N^{exact}$ (which is nonzero only at single point of the phase space) by the probability function over phase space, $f_N$,

$$f_N(\mathbf{q}_i, \mathbf{p}_i, t), \tag{4}$$

where the expression

$$f_N(\mathbf{q}_i, \mathbf{p}_i, t) \prod d\mathbf{q}_i d\mathbf{p}_i \tag{5}$$

determines the probability that the system will appear in the phase volume element $\prod d\mathbf{q}_i d\mathbf{p}_i$ around coordinates $\mathbf{q}_i$, $\mathbf{p}_i$.

The probability distribution function $f_N$ develops in time, according to the Liouville theorem, as

$$\frac{\partial f_N}{\partial t} + \sum_{i=1}^{i=N} [\frac{\partial f_N}{\partial \mathbf{q}_i} \frac{\partial H_N}{\partial \mathbf{p}_i} - \frac{\partial f_N}{\partial \mathbf{p}_i} \frac{\partial H_N}{\partial \mathbf{q}_i}] = 0. \tag{6}$$

Here the Hamiltonian $H_N$ describes the motion of particles and is given as

$$H_N = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \phi(\mathbf{q}_1, \mathbf{q}_2, ...\mathbf{q}_N, t), \tag{7}$$

where $m$ is the mass of a particle and $\phi$ is the potential energy.

Let us now define the reduced distribution function, $f_s$, as

$$f_s(\mathbf{q}_1, \mathbf{p}_1, \mathbf{q}_2, \mathbf{p}_2, ...\mathbf{q}_s, \mathbf{p}_s, t) = A_s \int f_N \prod_{i=s+1}^{N} d\mathbf{q}_i d\mathbf{p}_i \tag{8}$$

where $A_s$ is a normalization constant. Function $f_s$ describes the probability of finding the system of $s$ particles at points $\mathbf{q}_1$, ..., $\mathbf{q}_s$, $\mathbf{p}_1$, ... $\mathbf{p}_s$.

We can then obtain from (6)

$$\frac{\partial f_s}{\partial t} + \sum_{i=1}^{s} \frac{\mathbf{p}_i}{m} \frac{\partial f_s}{\partial \mathbf{q}_i} - \sum_{i=1}^{s} \sum_{j=1, j\neq i}^{s} \frac{\partial \phi_{ij}}{\partial \mathbf{q}_i} \frac{\partial f_s}{\partial \mathbf{p}_i} + \frac{1}{m} F_i^{ext} \frac{\partial f_s}{\partial p_i} -$$

tedy je jenom $f_{s+1}$, tj nejde to až do $N$

$$- \frac{(N-s)}{V} A_s \sum_{i=1}^{s} \int \frac{\partial \phi_{i,s+1}}{\partial \mathbf{q}_i} \frac{\partial f_{s+1}}{\partial p_i} d\mathbf{q}_{s+1} d\mathbf{p}_{s+1} = 0. \tag{9}$$

Here we have used the definition of $f_s$, we have assumed that $f_N$ vanishes at the boundary of the phase space, displaced to infinity, and we have used the fact that $f_N$ is a symmetric function of the coordinates of particles of the same kind; $V$ is the configuration-space volume of the system. Using further

$$A_s = \frac{1}{(N-s)!} \tag{10}$$

we finally obtain

$$L_s f_s = \frac{1}{m} \sum_{i=1}^{s} \int \frac{\partial \phi_{i,s+1}}{\partial \mathbf{q}_i} \frac{\partial f_{s+1}}{\partial \mathbf{p}_i} d\mathbf{q}_{s+1} d\mathbf{p}_{s+1}. \tag{11}$$

Here, $L_s$ is the Liouville operator for the $s$-particle distribution function; the right-hand side of the equation describes the effect of the other electrons.

The simplest - and most frequently used - chain of equations is given by the choice $s = 1$, namely

$$\frac{\partial f_1}{\partial t} + \mathbf{v}_1 \frac{\partial f_1}{\partial \mathbf{r}_1} + \frac{\mathbf{F}_1}{m} \frac{\partial f_1}{\partial \mathbf{v}_1} = \frac{1}{m} \int \frac{\partial \phi_{12}}{\partial \mathbf{r}_1} \frac{\partial P_{12}}{\partial \mathbf{v}_1} d\mathbf{r}_2 d\mathbf{v}_2. \tag{12}$$

where we have already used more common variables in velocity- space representation (v, r represention). Force $\mathbf{F}_1$ represents all external and field forces. The field (electrostatic) force is given by the average electric field, acting on one electron due to the effects of other electrons, as

$$- eE(\mathbf{r}_1, t) = -e \int \frac{\partial \phi_{12}}{\partial \mathbf{r}_1} f(2)\, d\mathbf{r}_2\, d\mathbf{v}_2 \tag{13}$$

where

$$f_1(\mathbf{r}_1, \mathbf{v}_1, t) = f(1) \tag{14}$$

$$f_2(\mathbf{r}_1, \mathbf{v}_1, \mathbf{r}_2, \mathbf{v}_2, t) = f(1) f(2) + P(1,2) \tag{15}$$

where $P$ is the pair correlation function. The foregoing expansion is called the *cluster expansion.* If $P = 0$, the two-particles distribution function is just the product of one-particle distribution functions; this means that particles 1 and 2 are uncorrelated. Thus $P$ is that part of $f_2$ which represents the *correlation* of the particles and is known as the pair correlation function.

Inserting this in the Eq. (12), we obtain

$$\frac{\partial f(1)}{\partial t} + \bar{v}_1 \frac{\partial f(1)}{\partial \bar{r}_1} + \frac{\bar{F}_1^{ext}}{m} \frac{\partial f(1)}{\partial \bar{v}_1} - \frac{1}{m} \frac{\partial f(1)}{\partial \bar{v}_1} \int \frac{\partial \phi_{12}}{\partial \bar{r}_1} f(2)\, d\bar{r}_2\, d\bar{v}_2 = \frac{1}{m} \int \frac{\partial \phi_{12}}{\partial \bar{r}_1} \frac{\partial P(1,2)}{\partial \bar{v}_1} d\bar{r}_2\, \partial \bar{v}_2. \quad (15)$$

The fourth term in the foregoing equation contains the average electric field experienced by one electron due to the other electrons and can be written as

$$-\frac{1}{m} \frac{\partial f(1)}{\partial \bar{v}_1} \cdot \int \frac{\partial \phi_{12}}{\partial \bar{r}_1} f(2)\, d\bar{r}_2\, d\bar{v}_2 = -\frac{e\bar{E}}{m} \cdot \frac{\partial f(1)}{\partial \bar{v}_1}$$

where the electric field is given as

$$(-e)\bar{E}(\bar{r}_1, t) = -\int \frac{\partial \phi_{12}}{\partial \bar{r}_1} f(2)\, d\bar{r}_2\, d\bar{v}_2$$

Consequently, the third and the fourth terms of the equation (15) can be written in the form

(*)

$$\frac{\bar{F}_1^{ext}}{m} \frac{\partial f(1)}{\partial \bar{v}_1} - \frac{1}{m} \frac{\partial f(1)}{\partial \bar{v}_1} \int \frac{\partial \phi_{12}}{\partial \bar{r}_1} f(2)\, d\bar{r}_2\, d\bar{v}_2 = \frac{\bar{F}}{m} \frac{\partial f(1)}{\partial \bar{v}_1}$$

where the term $\bar{F}$ represents all external and „field“ forces.

The right-hand side term in the foregoing equation (15), namely,

$$\frac{1}{m} \int \frac{\partial \phi_{12}}{\partial \bar{r}_1} \frac{\partial P(1,2)}{\partial \bar{v}_1} d\bar{r}_2\, \partial \bar{v}_2$$

must be therefore expressed by means of some approximation.

and where $P(1,2)$ is called the pair correlation function. (For an uncorrelated distribution of particles (1) and (2), $P(1,2) = 0$).

The foregoing forms (14), (15) point to one important property of this hierarchy, namely, that the system of equations is not closed. The equation for function $f_1$ contains function $f_2$, and so on. The system is not closed and some approximations are therefore required. This procedure will now be outlined.

The foregoing equation is usually written in the following form

$$\frac{\partial f_1}{\partial t} + \mathbf{v}\frac{\partial f_1}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m}\frac{\partial f_1}{\partial \mathbf{v}} = (\frac{\partial f}{\partial t})_c \tag{16}$$

where term $(\frac{\partial f}{\partial t})_c$ is called the collision term, or the collision integral

$$(\frac{\partial f}{\partial t})_c = \frac{1}{m}\int \frac{\partial \phi_{12}}{\partial \mathbf{r}}\frac{\partial P(1,2)}{\partial \mathbf{v}} d\mathbf{r}_2 d\mathbf{v}_2. \tag{17}$$

The equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m}\frac{\partial f}{\partial \mathbf{v}} = (\frac{\partial f}{\partial t})_c \tag{18}$$

is called the Boltzmann kinetic equation.

There exist several approaches which express the collision term explicitly. We shall mention the approach which fits the plasma physics theory well, the Fokker-Planck collision term. To penetrate at least partly into this rather complicated part of plasma physics theory, we shall first remind of some features of the theory of collisions of charged particles.

Let us assume that two charged particles with charges $e_1$, $e_2$ and masses $m_1$, $m_2$ are approaching each other (see Fig.1). For the sake of simplification let us suppose that the first particle is an electron and the second one is an ion. Due to its large mass, the ion can be considered as immovable. In this case, the geometry of this scattering is quite simple. Parameter $r_0$ is called the impact parameter, angle $\theta$ is called the scattering angle. For the electron with velocity $v$, mass $m_e$ and charge $e$ and for protons the scattering angle $\theta$, according to the classical theory of Rutherford, obeys

$$\cot(\frac{\theta}{2}) = \frac{1}{e^2}4\pi\epsilon_0 m_e v^2 r_0 \tag{19}$$

where $\epsilon_0$ is the permitivity of free space. The impact parameter for a $90^0$ scatter is

$$r_{0,90^0} = \frac{e^2}{4\pi\epsilon_0 m_e v^2}. \tag{20}$$

Due to the large amount of particles within a Debye sphere, the scattering process represents a cummulative effect of a large amount of weak deflections rather than the effect of a single close collision. The global effect can be considered as a random process with a set of small angle deflections $\triangle\theta$,

$$\triangle\,\theta = \frac{2e^2}{4\pi m_e v^2 r_0 \epsilon_0}. \tag{21}$$

The deflection depends on the impact parameter, $r_0$. Let us first consider the maximum and minimum impact parameters, $r_{0max}$, and $r_{0min}$. Defining the function $F(\Delta\theta)$ which describes the probability of the scattering in angle $\Delta\theta$, and expressing the number of scattering centers in a shell of the length $L$, radius $r_0$, and the thickness $dr_0$, the mean-square deflection, $<(\Delta\theta)^2>$, is

$$<(\Delta\theta)^2>=\int_{\Delta\theta_{min}}^{\Delta\theta_{max}}(\Delta\theta)^2F(\Delta\theta)d(\Delta\theta) \tag{22}$$

where $\Delta\theta_{max}$, $\Delta\theta_{min}$ correspond to impact parameters $r_{min}$, $r_{max}$, respectively. ($r_{min}$, $r_{max}$ will be determined later on). The mean-square deflection $<(\Delta\theta)^2>$ can be then determined as ($F(\Delta\theta)d(\Delta\theta)=nL2\pi r_0dr_0$)

$$<(\Delta\theta)^2>=\frac{1}{2\pi}\frac{nLe^4}{\epsilon_0^2m_e^2v^4}\int_{r_{0min}}^{r_{0max}}\frac{dr_0}{r_0}=\frac{1}{2\pi}\frac{nLe^4}{\epsilon_0^2m_e^2v^4}\ln\frac{r_{0max}}{r_{0min}} \tag{23}$$

and where $L$ is the distance traversed by the particle.

This expression diverges for $r_{0max}\rightarrow\infty$, and for $r_{0min}\rightarrow 0$. The first limit corresponds to the existence of the long-range Coulomb potential $\phi$

$$\phi=\frac{1}{4\pi\epsilon_0}\frac{e^2}{\mathbf{r}}. \tag{24}$$

The maximum value of $r_0$, $r_{0max}$, can be determined by the effect of Debye shielding. In a plasma with density $n$, the perturbation of the electron density (on the background of immobile heavy ions) causes the perturbation of potential $\phi$. This potential depends on space (radial) coordinate $r$, the distance from the perturbation, as

$$\phi=\phi_0\exp(-\frac{r}{\lambda_D}). \tag{25}$$

Quantity $\lambda_D$, the Debye length, is given as

$$\lambda_D=\sqrt{\frac{\epsilon_0kT_e}{ne^2}} \tag{26}$$

where $kT_e$ is the thermal energy of electrons. The shielding is caused by a cload of particles of the opposite sign (in our case, of ions) close to the locality of the perturbation. The Debye length is then a measure of the effect of the potential perturbation. It is therefore quite natural to take $r_{0max}=\lambda_D$. The second limit, $r_{0min}$, is taken as the impact parameter for the deflection of $90^0$. Accordingly, the expression for $<(\Delta\theta)^2>$ is

$$<(\Delta\theta)^2>=\frac{1}{2\pi}\frac{nLe^4}{m_e^2v^4\epsilon_0^2}\ln\Lambda \tag{27}$$

where

$$\Lambda = \frac{\lambda_D}{r_{0min}}; \quad r_{0min} \approx \frac{e^2}{3kT_e\epsilon_0}. \tag{28}$$

It is then possible to estimate the cross-section $\sigma_{90^0M}$ for $90^0$ multiple scattering as

$$\sigma_{90^0M} = \frac{1}{nL_{90^0}} = \frac{1}{2\pi}\frac{e^4}{m_e^2v^4\epsilon_0^2}\ln\Lambda. \tag{29}$$

Ratio $\frac{\sigma_{90^0M}}{\sigma_{90^0S}}$ (where $\sigma_{90^0S}$ is the cross-section of the single scattering) is

$$\frac{\sigma_{90^0M}}{\sigma_{90^0S}} \approx 8\ln\Lambda. \tag{30}$$

Usually for laboratory (and thermonuclear) plasmas $\ln\Lambda \approx 20$. Consequently, the effect of small- angle scattering exceeds the effect of large, single scattering.

Let us return to our collisional kinetic equation

$$\frac{\partial f}{\partial t} + \mathrm{v}\frac{\partial f}{\partial \mathrm{r}} + \frac{\mathbf{F}}{m}\frac{\partial f}{\partial \mathrm{v}} = (\frac{\partial f}{\partial t})_c, \tag{31}$$

where, generally,

$$(\frac{\partial f}{\partial t})_c = \frac{1}{m}\int \frac{\partial \phi_{12}}{\partial \mathrm{r}}\frac{\partial P_{1,2}}{\partial \mathrm{v}}\, d\mathrm{r}_2 d\mathrm{v}_2. \tag{32}$$

Our aim is now to find the explicit form of collision integral (32). Usually, two ways to do this are mentioned. The first one consists in finding the explicit form of the correlation function and, simultaneously, in finding an approximative procedure, enabling closing of the chain of the BBGKY hierarchy. This approach includes - inspite of its exactness - a lot of difficult problems. The second way, which is usually used, in fact omits the discussion of the hierarchy and consists in expressing the collision integral approximately as a function of distribution function $f$. This approach is based on the Fokker-Planck equation; its basic features will now be described.

Collision term $(\frac{\partial f}{\partial t})_c$ can be expressed as the change of distribution function $f$ over short time element $\triangle t$,

$$(\frac{\partial f}{\partial t})_c = \frac{1}{\triangle t}[f(\mathrm{x}, \mathrm{v}, t + \triangle t) - f(\mathrm{x}, \mathrm{v}, t)]. \tag{33}$$

Using probability function $\psi(\mathrm{v}, \triangle t)$, describing the probability with which a particle with velocity v, will change its velocity to $\mathrm{v} + \triangle\mathrm{v}$ during time element $\triangle t$, the change in $f$ can be expressed as

$$f(\mathrm{x}, \mathrm{v}, t + \triangle t) = \int f(\mathrm{x}, \mathrm{v} - \triangle\mathrm{v}, t)\psi(\mathrm{v} - \triangle\mathrm{v}, \triangle\mathrm{v})d(\triangle\mathrm{v}). \tag{34}$$

(Since function $\psi$ is time-independent, the corresponding probability process is a Markoff process).

Let us expand the integrand as

$$f(\mathbf{x}, \mathbf{v}-\Delta\mathbf{v}, t)\psi(\mathbf{v}-\Delta\mathbf{v}) = f(\mathbf{x}, \mathbf{v}, t)\psi(\mathbf{v}, \Delta\mathbf{v}) - \sum_i \frac{\partial(f\psi)}{\partial v_i} \Delta v_i + + \frac{1}{2}\sum_{i,j} \frac{\partial^2(f\psi)}{\partial v_i \partial v_j} \Delta v_i \Delta v_j. \quad (35)$$

Prescribing, as usually that the total probability satisfies

$$\int \psi(\mathbf{v}, \Delta\mathbf{v}) d(\Delta\mathbf{v}) = 1 \quad (36)$$

and defining the Fokker-Planck coefficients as

$$< \frac{\Delta v_i}{\Delta t} >= \int \psi \Delta v_i \, d(\Delta\mathbf{v}) \frac{1}{\Delta t} \quad (37)$$

$$< \frac{\Delta v_i \Delta v_j}{\Delta t} >= \int \psi \Delta v_i \Delta v_j \, d(\Delta\mathbf{v}) \frac{1}{\Delta t}, \quad (38)$$

we can express the term $(\frac{\partial f}{\partial t})_c$ as

$$(\frac{\partial f}{\partial t})_c = -\sum_i \frac{\partial(< \frac{\Delta v_i}{\Delta t} > f)}{\partial v_i} + \frac{1}{2}\sum_{i,j} \frac{\partial^2(< \frac{\Delta v_i \Delta v_j}{\Delta t} > f)}{\partial v_i \partial v_j}. \quad (39)$$

Term $< \frac{\Delta v_i}{\Delta t} >$ is called the dynamic friction term, term $< \frac{\Delta v_i \Delta v_j}{\Delta t} >$ is a component of the diffusion tensor.

The explicit calculation of the friction term and of the components of the diffusion tensor enables the collision term to be expressed for the case of the single- charge ion background [4] as

$$(\frac{\partial f}{\partial t})_c = \sum_n \frac{e^4 \ln \Lambda}{4\pi\epsilon_0^2 m^2} [ - \frac{\partial(\frac{\partial H_n(\mathbf{v})}{\partial v_i} f(\mathbf{v}))}{\partial v_i} + \frac{1}{2} \frac{\partial^2(\frac{\partial^2 G_n(\mathbf{v})}{\partial v_i \partial v_j} f(\mathbf{v}))}{\partial v_i \partial v_j} ] \quad (40)$$

where

$$H_n(\mathbf{v}) = (1 + \frac{m}{m_n}) \int \frac{f_n(\mathbf{v}')}{|\mathbf{v}-\mathbf{v}'|} \, d\mathbf{v}' \quad (41)$$

$$G_n(\mathbf{v}) = \int f_n(\mathbf{v}') \left|\mathbf{v}-\mathbf{v}'\right| d\mathbf{v}'. \quad (42)$$

Here, $m$ is the mass of a scattering (test) particle, $m_n$ the mass of a scattered particle; $n$ labels the particle species. Let us consider the scattering particles under $\delta$-function velocity distribution [2]

$$f(\mathbf{v}_1) = a\delta(\mathbf{v} - \mathbf{U}) \quad (43)$$

and the scattered (field) particles under Maxwellian distribution $f(\mathrm{v}_2)$

$$f(\mathrm{v}_2) = f_M(\mathrm{v}_2) = \frac{na^3}{\pi^{\frac{3}{2}}}\exp(-a^2 v_2^2); \quad a^2 = \frac{m_2}{2kT}. \tag{44}$$

Let us define deflection time $\tau_D$ as

$$\tau^D = \frac{U^2}{\frac{\partial U_1^2}{\partial t}} \tag{45}$$

where $\tau^D$ is an estimate of the time, required to achieve isotropy. Considering the interaction of electrons with protons, the foregoing procedure yields

$$\tau_{ei}^D = \frac{2\pi\epsilon_0^2 m_e^{\frac{1}{2}}(2kT_e)^{\frac{3}{2}}}{ne^4 \ln\Lambda} \tag{46}$$

Labelling $\tau_{ei}^D = \tau_{ei}$ and taking quantity $\nu_{ei} = \frac{1}{\tau_{ei}}$ to be the electron-ion collision frequency, we see that $\nu_{ei}$ decreases with inreasing temperature. Considering, e.g., the thermonuclear parameters ($n = 10^{20}m^{-3}$, $kT_e = 20keV$), and electron-proton plasma, we obtain $\nu_{ei} \approx 1.8 \times 10^3 s^{-1}$. The frequencies of waves, used for plasma heating, are in the range $\nu \approx 10^6 s^{-1} - 10^{12} s^{-1}$; in these cases, the effect of the collision frequency is usually neglected.

In rough approximation, the collision term

$$(\frac{\partial f}{\partial t})_c \approx (f - f_M)\nu \tag{47}$$

where $f_M$ is the Maxwell distribution and $\nu$ is the collision frequency. This so-called Bhatnagar, Gross and Krook model fits collisional phenomena in weakly ionized plasmas. For a fully ionized plasma, the foregoing expression requires [3] some modification. A special form of the collision term, given by the Lenard-Bernstein model, is presented in Section 4.

## 2. THE VLASOV EQUATION AND LANDAU DAMPING

The Boltzmann equation with its collision term describes the plasma behaviour in a broad region of plasma parameters. Nevertheless, in many applications, the collision term can be neglected. For this regime, the necessary conditions can be defined as

$$\omega \gg \nu_{ei}; \quad \omega_{pe} \gg \nu_{ei}. \tag{48}$$

Here $\omega$ is the frequency of the wave, whose propagation in plasma is being investigated, and $\omega_{pe}$ is the plasma frequency,

$$\omega_{pe}^2 = \frac{ne^2}{\epsilon_0 m_e} \tag{49}$$

where $n$ is the plasma density.

Neglecting the collision term, we obtain the Vlasov equation

$$\frac{\partial f}{\partial t} + \frac{\mathrm{dr}}{\mathrm{d}t}\frac{\partial f}{\partial \mathrm{r}} + \frac{\mathbf{F}}{m}\frac{\partial f}{\partial \mathrm{v}} = 0. \tag{50}$$

where $\mathbf{F}$ is the averaged force, acting on particles. Considering $\mathbf{F}$ to be the Lorentz force,

$$\mathbf{F} = q[\mathbf{E} + \mathrm{v} \times \mathbf{B}] \tag{51}$$

we obtain the equation, referred to as the Vlasov equation

$$\frac{\partial f}{\partial t} + \frac{\mathrm{dr}}{\mathrm{d}t}\frac{\partial f}{\partial \mathrm{r}} + \frac{q}{m}[\mathbf{E} + \mathrm{v} \times \mathbf{B}]\frac{\partial f}{\partial \mathrm{v}} = 0. \tag{52}$$

Here, $\mathbf{E}$ and $\mathbf{B}$ are the averaged electric and magnetic fields, respectively.

The Vlasov equation is the equation, used most frequently in discussing different features of the plasma kinetic theory, especially in high-temperature plasma physics.

It is worthwile mentioning some important properties of this equation. The first one follows from the fact that the collision term has been neglected. The equation, therefore, has the form of the continuity equation in phase space, analogous to the Liouville equation. The time derivative of $f$, $\frac{\mathrm{d}f}{\mathrm{d}t}$, is zero along the phase space trajectory, given by

$$\frac{\mathrm{dr}}{\mathrm{d}t} = \mathrm{v}; \quad \frac{\mathrm{dv}}{\mathrm{d}t} = \frac{q}{m}[\mathbf{E} + \mathrm{v} \times \mathbf{B}]. \tag{53}$$

Consequently, the density of particles in phase space during their motion along their trajectories does not change. (On the contrary, term $(\frac{\partial f}{\partial t})_c \neq 0$ changes the particle density on the phase space trajectory).

The second important property is given by the fact [1] that trajectories (53) are the characteristics of partial differential equation (52). Consequently, any function of the constants of motion is the solution of the Vlasov equation. (For example, if $\mathbf{E} = \mathbf{B} = 0$, the total kinetic energy is a constant of the motion. Therefore, $f = f(\frac{1}{2}mv^2)$, and, e.g., $f = \exp(\frac{mv^2}{2kT})$ are also solutions of the Vlasov equation).

The Vlasov equation depends on the electric and magnetic fields $\mathbf{E}$, $\mathbf{B}$, and these quantities are given by charge densities and by currents, flowing through the plasma. It is, therefore, necessary to couple the Vlasov equation with the system of Maxwell equations

$$\nabla\cdot\mathbf{E}(\mathrm{x}, t) = \sum_\alpha \frac{q_\alpha}{\epsilon_0}\int f_\alpha(\mathrm{x}, \mathrm{v}, t) d^3\mathrm{v} \tag{54}$$

$$\nabla\times\mathbf{H} = \sum_\alpha q_\alpha \int \mathrm{v} f_\alpha(\mathrm{x}, \mathrm{v}, t) d^3\mathrm{v} + \epsilon_0\frac{\partial \mathbf{E}}{\partial t} \tag{55}$$

$$\nabla\cdot\mathbf{B} = 0 \tag{56}$$

$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{57}$$

$$\mathbf{B} = \mu\mu_0\mathbf{H}. \tag{58}$$

Expressing charge $\rho$ and current $\mathbf{j}$ densities as

$$\rho = \sum_\alpha \rho_\alpha \tag{59}$$

$$\mathbf{j} = \sum_\alpha q_\alpha \int \mathbf{v} f_\alpha d^3\mathbf{v}, \tag{60}$$

defining density $n_\alpha$ as

$$n_\alpha = \int f_\alpha d^3\mathbf{v} \tag{61}$$

and total charge $\rho$ as

$$\rho = \sum_\alpha q_\alpha \int f_\alpha d^3\mathbf{v} \tag{62}$$

we obtain

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{63}$$

$$\nabla \times \mathbf{H} = \mathbf{j} + \epsilon_0 \frac{\partial E}{\partial t} \tag{64}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{65}$$

The necessity of this selfconsistent solution causes the main difficulties in the plasma kinetic theory. The problem itself is nonlinear, and it is necessary to solve a set of coupled partial differential equations. Moreover, waves, which can appear in a plasma have - just due to this coupling - a rather complicated polarization, in many cases quite unlike the simple polarization of an electromagnetic wave, propagating in vacuum.

Due to these complications, we shall confine ourselves to the following simplest case. We shall assume a spatially homogeneous plasma without external electric and magnetic fields. We shall assume that ions form an uniform immobile background, and discuss only the changes of the electron distribution function. (The concept of the ion's immobility - with regard to the electron component - is justified by the large ratio $\frac{m_i}{m_e}$). Further, we shall assume that the perturbation of the electron distribution function , generating the space charge, will be only one-dimensional, that there will be no fluctuating magnetic field and, consequently, that the discussed waves will only have the electric component. These assumptions enable us to choose, from the set of waves possibly existing in plasmas, the simplest case - the electrostatic (longitudinal, Langmuir, potential) waves; the electric field of these waves can be derived from the potential function. Under these assumptions, the electric field of the wave, $\mathbf{E}$, satisfies $\mathbf{E} \parallel \mathbf{k}$, and the perturbation of the velocity, $\mathbf{v}$ satisfies $\mathbf{v} \parallel \mathbf{k}$, where $\mathbf{k}$ is the wave vector.

The set of coupled equations, describing the evolution of the perturbation of the electron distribution function, can then be simplified to the following pair of equations (considering $x$ to be the space coordinate, $\mathbf{x} \parallel \mathbf{E}$ and $v_x \equiv v$ the velocity coordinate)

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{eE}{m}\frac{\partial f}{\partial v} = 0 \tag{66}$$

$$\nabla\cdot\mathbf{E} = -\frac{e}{\epsilon_0}\int f d^3\mathrm{v} \tag{67}$$

Here, the first equation is the one-dimensional Vlasov equation, the second equation is the Poisson equation.

The system is obviously nonlinear, and it is necessary to use some perturbation approach. Considering only small perturbations, we can express the distribution function as the sum of the unperturbed part $f_0 = f_0(v)$ and of the perturbation $f_1(x, v, t)$

$$f = f_0 + f_1; \quad f_1 \ll f_0. \tag{68}$$

Perturbation $f_1$ depends not only on the velocity, but also on coordinate $x$ and on time $t$. The unperturbed part is time-independent, and because the plasma is supposed to be homogeneous, $f_0$ is also space-independent. Therefore, $f_0$ generates no electric field, $E_0 = 0$.

Consequently, our linearized system of the Vlasov and Poisson equations reads

$$\frac{\partial f_1}{\partial t} + \mathrm{v}\frac{\partial f_1}{\partial \mathrm{x}} - \frac{e}{m}\mathbf{E}_1\frac{\partial f_0}{\partial \mathrm{v}} = 0 \tag{69}$$

$$\nabla\cdot\mathbf{E}_1 = -\frac{e}{\epsilon_0}\int f_1 d^3\mathrm{v}. \tag{70}$$

This system was originally solved by Vlasov [5]. Nevertheless, inspite of the apparent simplicity of these equation, the complete and correct solution has been given only by Landau [6]. Landau damping is perhaps the most important phenomenon of high-temperature plasma physics.

The usual approach starts with the application of Fourier analysis and of Laplace transforms. Let us first Fourier-analyze both equations (in what follovs we shall omit the vector labelling). Using

$$f_1(\mathrm{k}, \mathrm{v}, t) = \int f_1(\mathrm{x}, \mathrm{v}, t)e^{-i\mathrm{kx}}d\mathrm{x} \tag{71}$$

$$\mathrm{E}_1(\mathrm{k}, t) = \int \mathrm{E}_1(\mathrm{x}, t)e^{-i\mathrm{kx}}d\mathrm{x} \tag{72}$$

we obtain

$$\frac{\partial f_1(\mathrm{k}, \mathrm{v}, t)}{\partial t} + i\mathrm{kv}f_1(\mathrm{k}, \mathrm{v}, t) - \frac{e\mathrm{E}_1(\mathrm{k}, t)}{m}\frac{\partial f_0(\mathrm{v})}{\partial \mathrm{v}} = 0. \tag{73}$$

Let us define [1]

$$F_0(u) = \int f_0(\mathbf{v})\, \delta(u - \mathbf{k}\cdot\frac{\mathbf{v}}{k})\, d\mathbf{v} \tag{74}$$

$$F_1(k,u,t) = \int f_1(\mathbf{k},\mathbf{v},t)\, \delta(u - \mathbf{k}\cdot\frac{\mathbf{v}}{k})\, d\mathbf{v}. \tag{75}$$

We then convert (73) to read

$$\frac{\partial F_1(k,u,t)}{\partial t} + ikF_1(k,u,t) - \frac{e}{m}E_1(k,t)\frac{\partial F_0(u)}{\partial u} = 0 \tag{76}$$

with

$$E_1(k,t) = \frac{ie}{\epsilon_0 k}\int F_1(k,u,t)du. \tag{77}$$

Let us now apply the Laplace transform

$$F_1(k,u,p) = \int_0^\infty F_1(k,u,t)e^{-pt}dt \tag{78}$$

$$E_1(k,p) = \int_0^\infty E_1(k,t)e^{-pt}dt \tag{79}$$

where $Re(p) > x_0$, and where $E_1$ and $F_1$ are assumed to have the form $e^{x_0 t}$ for $t > 0$. The Laplace transform of (76) reads

$$(p+iku)F_1(k,u,p) - \frac{eE_1(k,p)}{m}\frac{\partial F_0(u)}{\partial u} = F_1(k,u,t=0). \tag{80}$$

Using (70), (77-80), we can express $E_1(k,p)$ as

$$E_1(k,p) = \frac{ie}{\epsilon_0 k}\int F_1(k,u,p)\, du = \frac{ie}{\epsilon_0 k D(k,p)}\int \frac{F_1(k,u,t=0)}{p+iku}\, du \tag{81}$$

where

$$D(k,p) = 1 - \frac{ie^2}{\epsilon_0 mk}\int \frac{\partial F_0(u)}{\partial u}\frac{1}{p+iku}\, du. \tag{82}$$

Function $D$ is called the plasma dielectric function.

The expression for $E_1(k,t)$ will be determined by means of the inverse Laplace transform

$$E_1(k,t) = \frac{1}{2i\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} E(k,p)\, e^{pt}\, dp. \tag{83}$$

While the approach, leading to the expression for $E_1(k,p)$ creates no problems, the inverse Laplace transform requires the discussion of the singularities, appearing in the foregoing expression. Landau found that its careful analysis yields an important correction of the original Vlasov solution. A thorough discussion of the properties of the form of $E_1(k,p)$ (81) indicates that, under the assumption that $\frac{\partial F_0}{\partial u}$, $F_1$ are analytic functions of $u$, the only singularities which determine the time asymptotic solution of $E_1(k,t)$ are poles defined as

$$D = 0 \tag{84}$$

or

$$1 - \frac{ie^2}{\epsilon_0 mk} \int \frac{\frac{\partial F_0}{\partial u}}{p + iku}\, du = 0. \tag{85}$$

Labelling these poles as $p_j$ and the residues of $E_1(k,p)$ at these poles as $R_j$, for a sufficiently large time $t$, the asymptotic form of $E_1(k,t)$ can then be expressed as

$$\lim_{t\to\infty} E_1(k,t) = \sum_j R_j e^{p_j t}. \tag{86}$$

Expressing further $p_j$ as

$$p_j(k) = -i\omega(k)_j - \gamma(k)_j, \tag{87}$$

the time-asymptotic form of electric field $E_1(k,t)$ becomes

$$\lim_{t\to\infty} E_1(k,t) = \sum_j R_j e^{-i\omega(k)_j t - \gamma(k)_j t}. \tag{88}$$

For $\gamma_j > 0$, the oscillations are asymptotically damped. On the contrary, for $\gamma_j < 0$, an instability appears.

Frequencies $\omega_j$ and decrements $\gamma_j$ are determined by the dispersion equation

$$D(k,p) = 0. \tag{89}$$

Landau found the correct way of integrating of (85).

Let us first discuss the dispersion relation

$$1 - \frac{ie^2}{\epsilon_0 mk} \int \frac{\partial F_0}{\partial u} \frac{1}{p + iku}\, du = 0 \tag{90}$$

for $k \approx 0$, i.e. for the infinite wavelength limit. In this case we obtain from (90) the dispersion equation valid for this approximation

$$D = 1 + \frac{\omega_p^2}{p^2} = 0 \tag{91}$$

with the solution

$$p = \pm i\omega_p. \tag{92}$$

Consequently, in this lowest approximation, the plasma oscillations are undamped with the basic frequency

$$\omega = \omega_p = \sqrt{\frac{ne^2}{\epsilon_0 m}}. \tag{93}$$

Integrating integral in (85) by parts, assuming that $\frac{iku}{p} \ll p$ and expanding fraction $\frac{1}{p+iku}$, we obtain

$$1 + \frac{e^2}{\epsilon_0 m p^2} \int F_0(u)[1 - \frac{2iku}{p} - \frac{3k^2u^2}{p^2} - ...]\, du. \tag{94}$$

Taking

$$p = \pm i\omega_p + i\omega^{(1)} \tag{95}$$

as the first approximation and considering $F_0(u)$ to be Maxwellian, we obtain

$$\omega^{(1)} = \pm\omega_p \frac{3}{2} k^2 \lambda_D^2. \tag{96}$$

Parameter $\lambda_D$ is the Debye length

$$\lambda_D = \sqrt{\frac{\epsilon_0 k T_e}{n e^2}}. \tag{97}$$

Whereas solution (93) corresponds to simple plasma oscillations, solution (95) describes the dispersion relation of plasma (longitudinal, electrostatic, Langmuir) waves

$$\omega(k) = \pm\omega_p(1 + \frac{3}{2}k^2\lambda^2). \tag{98}$$

Nevertheless, expansion (94) provides no possibility of finding the damping rate, $\gamma$. For this purpose, it is necessary to discuss the full expression (90). The integration path crosses the pole at $u = \frac{ip}{k}$. Vlasov neglected this resonance pole and only considered the principal part of integral (90), leading to the foregoing dispersion. Landau, who carried out the integration properly, included this resonance effect. Since $E(k,p)$ in (81) was originally defined only for $Rep > 0$, it was necessary to find the correct analytic continuation of the integral in dispersion relation (90) also for $Rep < 0$. Landau found the proper way of integration along a special path (now called the Landau countour - see Fig. 2), passing half-way around and below the pole. The proper integration enables (90) to be expressed as

$$1 = \frac{\omega_p^2}{k^2}\,[\,P\int_{-\infty}^{+\infty} \frac{\partial F_0}{\partial u}\frac{1}{u - \frac{\omega}{k}}\,du + i\pi\frac{\partial F_0}{\partial u}/u = \frac{\omega}{k}\,], \tag{99}$$

where $P$ is the principal part of the integral. Considering $F_0$ to be the Maxwellian distribution ($K$ is the Boltzmann constant, $T_e$ is the electron temperature)

$$F_0(u) = \sqrt{\frac{m}{2\pi K T_e}}\, e^{-\frac{mu^2}{2KT_e}} \tag{100}$$

we obtain the dispersion relation in the form

$$1 = -\frac{\omega_p^2}{p^2}[1 - \frac{3k^2}{p^2}\frac{KT_e}{m}] + \sqrt{\frac{\pi}{2}}(\frac{m}{KT_e})^{\frac{3}{2}}\frac{\omega_p^2\, p}{k^2}\exp(\frac{mp^2}{2k^2KT_e}). \tag{101}$$

Using definition (87), the Landau damping decrement $\gamma$ comes out as

$$\gamma = \sqrt{\frac{\pi}{8}}\frac{\omega_p}{(k\lambda_D)^3}\exp[-\frac{1}{2(k\ \lambda_D)^3} - \frac{3}{2}]. \tag{102}$$

Reverting to expressions (86) and (87), the positive sign of $\gamma$ expresses the damping. Consequently, Langmuir waves, which penetrate through the plasma with Maxwellian velocity distribution are always damped. According to (99) and to the Maxwellian distribution in Fig. 3, the damping corresponds to the negative slope of the distribution function. On the contrary, the distribution with the positive slope can cause an instability. An example of such a distribution (the so called bump-in-tail distribution) is shown in Fig. 4; this distribution can be realized by a warm beam of electrons, which propagate through the Maxwellian plasma and which has average velocity $u_B$. The instability can be fed by the group of electrons, which form the positive slope of the distribution function.

Let us now return to the case of damping. Since the damping is the effect of the singularity

$$\omega - kv = 0, \tag{103}$$

the damping is caused by the interaction of waves with resonant particles, i.e. with particles whose velocity is equal to the phase velocity of the wave. These particles are, therefore, able to exchange their energy with the wave.

This is the key mechanism of the wave-particle interaction. Nevertheless, this mechanism itself does not explain the damping effect. For the physical explanation, several models have been proposed. Most of them overlap in the following picture.

Let us consider an electron, whose velocity is slightly higher than the phase velocity of the wave. This particle will be decelerated by the wave, and will, therefore, impart a part of its kinetic energy to the wave. A particle, moving slower, will draw energy from the wave. The total energy balance and, therefore, the damping or the instability of the wave, will depend on the number of particles with velocities lower and higher than the phase velocity of the wave. Obviously, if there is a larger number of particles moving slower than the amount of particles moving faster, the net energy balance signals that the wave must be damped. This is exactly the case of Landau damping, discussed for the Maxwellian distribution and presented in Fig. 3.

It is therefore possible to conclude that the opposite case, i.e. the case with the positive slope of the distribution function, will cause an instability, energy being transmitted from electrons to the wave. This case is outlined in Fig. 4 for the bump-on-tail instability, where obviously an excess of resonant particles with higher velocity causes the instability. This effect is well based theoretically, and has been verified by a set of different types of experiments.

## 3. TRAPPED PARTICLES AND THEIR INFLUENCE ON LANDAU DAMPING

The linearization of the Vlasov and Maxwell equations requires an infinitesimally low perturbation(and, therefore, an infinitesimally low amplitude of the wave). In this case, the velocity of the particle, which interacts with the perturbation, can be considered as unaffected by the perturbation and therefore constant. In fact, it is necessary to expect the perturbation to have a finite level and, therefore, the amplitude of the corresponding wave to be finite, too. The wave with the finite amplitude will cause a change of the particle velocity, and, consequently, the mechanism of Landau damping will cease to be linear. This will result in the appearance of a new phenomenon - particle trapping. This effect is not only of basic importance for the validity of Landau damping concept, but naturally creates a cause for the saturation of important instabilities. Moreover, the trapped particles represent a typical case of a nonlinear oscillator. The model of the nonlinear oscillator forms a key paradigma for the discussion of the deterministic chaos in Hamiltonian systems, and has important consequences for the validity of the quasilinear approximation of wave-particle interaction, as will be discussed later on. We shall, therefore, mention this effect more thoroughly.

Let us consider a monochromatic electrostatic (Langmuir) wave with potential $\varphi$,

$$\varphi = \varphi_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi), \tag{104}$$

where $\omega$, $\mathbf{k}$ are the frequency and the wave vector of the wave, respectively, and where $\mathbf{r}$ is the space vector. Let us consider the simplest case of a homogeneous plasma without magnetic field, and let us identify the direction of the wave propagation with the direction of the $x$-coordinate of the Cartesian system. The Hamiltonian of a particle in the field of this wave reads

$$H_x = \frac{1}{2m} p_x^2 - e\varphi_0 cos(kx - \omega t) \tag{105}$$

with $\phi = 0$. Variables $x$, $p_x$ are the canonically conjugated coordinate and momentum, respectively.

Let us use canonical transformations with generating functions $F_2^{(1)}$, $F_2^{(2)}$ (for symbolics, see e.g.[7])

$$F_2^{(1)} = \frac{P^{(1)}}{k}(kx - \omega t) + x\, m\, \frac{\omega}{k} \tag{106}$$

$$F_2^{(2)} = PQ^{(1)} - \frac{m\omega^2}{2k^2}t, \tag{107}$$

and let us apply them to the foregoing Hamiltonian. We then obtain the new Hamiltonian in the form

$$H = \frac{1}{2m}P^2 - e\varphi_0 \cos kQ \tag{108}$$

where

$$p_x = P + m\frac{\omega}{k}; \quad x = Q + \frac{\omega}{k}t. \tag{109}$$

This Hamiltonian is identical with the Hamiltonian of the mathematical pendulum. For

$$H < e\varphi_0 \tag{110}$$

both coordinates $Q$, $P$ oscillate. Let us label the amplitude of oscillations as $Q_{max}$. Then for $kQ_{max} \ll 1$, the foregoing Hamiltonian describes the harmonic oscillations with the equation

$$\frac{\mathrm{d}^2 Q}{\mathrm{d}t^2} + \omega_{B0}^2 Q = 0 \tag{111}$$

where

$$\omega_{B0} = \sqrt{\frac{e\varphi_0 k^2}{m}} \tag{112}$$

is the angular frequency of the harmonic oscillations, of oscillations of the so-called well trapped particles. For generally trapped (but still librating) particles, the frequency of oscillations depends on the energy of the oscillations,

$$\omega_B = \omega_{B0}\frac{\pi}{2}K(\mathrm{k})^{-1}, \tag{113}$$

where

$$2\mathrm{k}^2 = 1 + \frac{\mathrm{H}}{\mathrm{e}\varphi_0} \tag{114}$$

and where $K$ is the total elliptic integral of the first kind.

The well known phase space picture of trajectories of particles with the dynamics, described by Hamiltonian (108) is presented in Fig. 5. Whereas the trapped (oscillating, librating) particles form closed trajectories, the untrapped particles form unclosed $P$-$Q$ trajectories. The untrapped and trapped particles are separated by the separatrix, crossing the $x$-coordinate axis.

Let us now return to our problem of Landau damping. As has already been said, one of the basic assumptions of the analysis consists in the requirement that $v = const$, which can be exactly satisfied only for an infinitesimally small wave amplitude. For a finite amplitude, the particles start to oscillate, as has just been shown. Therefore, for a finite amplitude, the Landau procedure is valid in the time interval $\Delta t$, for which the change of $v$ can be considered negligible. This interval can be estimated using trapped particle dynamics. The period of the well trapped particle oscillations, $\tau_{osc}$,

$$\tau_{osc} = \frac{2\pi}{\omega_{B0}} = 2\pi\sqrt{\frac{m}{e\varphi_0 k^2}}, \tag{115}$$

is considered to be a well founded measure of the validity of the Landau damping procedure. This procedure is obviously valid under the assumption that the wave is damped during a period shorter than $\tau_{osc}$. The condition for this is usually

$$\gamma_L \tau_{osc} \gg 1 \tag{116}$$

where $\gamma_L$ is the Landau damping rate (Landau decrement). For $\gamma_L \tau_{osc} < 1$, the nonlinear effect will be achieved even before a substantial part of the wave energy is absorbed.

The effect of trapping on Landau damping was discussed by O'Neil [8]. Using an analytical approach, based on the carefully taking into account the effect of generally trapped and untrapped particles (briefly mentioned above), O'Neil found that the expression for Landau damping in the case of waves with a finite amplitude is applicable at the beginning of the interaction, for times shorter than $\tau_{osc}$. The actual damping rate tends to decrease asymptotically to zero. Beside this damping, the wave amplitude oscillates with a frequency close to $\omega_{B0}$. For the instantaneous value of the wave decrement, $\gamma(t)$, O'Neil found the following estimate

$$\int_0^\infty \gamma(t)dt \approx \tau_{osc}\gamma_L. \tag{117}$$

This expression can, therefore, be used to estimate the total amount of energy of finite amplitude waves, absorbed in a collisionless plasma. The motion of trapped particles is shown in Fig. 6 (the complicated evolution is caused by the nonlinearity of the frequency of trapped particles). The oscillation of the wave amplitude $\varphi$ is also shown in Fig. 6.

Let us now shortly mention the Vlasov theory of small-amplitude waves, which propagate in a plasma in an external uniform magnetic field. The orbits of particles are due to their gyration rather complicated; the corresponding wave-plasma interaction is then described by a complicated form of the dispersion relation. For the general form of the dispersion relation, see, e.g. [3] or [2]. In this chapter, we shall give only an outline of this derivation of the dispersion relation for the electrostatic waves, propagating in a magnetized plasma; we shall follow [9].

Let us suppose that the electrostatic wave has its potential $\varphi_1$

$$\varphi_1 = \varphi_{10}\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t) \tag{118}$$

and let the wave vector $\mathbf{k}$ has the perpendicular and the parallel component,

$$\mathbf{k} = k_y \mathbf{y}_0 + k_{\|}\mathbf{z}_0 \tag{119}$$

(for simplicity, we express the perpendicular component of $\mathbf{k}$ by the component $k_y$). Let the homogeneous magnetic field $\mathbf{B} = B_0\mathbf{z}_0$ has its field lines parallel to the $z$-coordinate. Let us express, as usually, the distribution function $f$ for electrons in the form

$$f(\mathbf{r},\mathbf{v},t) = f_0(\mathbf{v}) + f_1(\mathbf{r},\mathbf{v},t) \tag{120}$$

where $f_0$ is the unperturbed part of $f$ and $f_1$ is the perturbation. The Vlasov equation for $f_1$ takes in this case the form

$$\frac{\partial f_1}{\partial t} + \mathbf{v}\frac{\partial f_1}{\partial \mathbf{r}} - \frac{e}{m}\mathbf{v}\times\mathbf{B}\,\frac{\partial f_1}{\partial \mathbf{v}} = -\frac{e}{m}\,\nabla\varphi_1\frac{\partial f_0}{\partial \mathbf{v}}. \tag{121}$$

Using the methods of characteristics (LHS part of Eq. (121) can be written in the form of the total time derivative, following the unperturbed trajectory of the particle in the $\mathbf{r}$, $\mathbf{v}$ space), the solution of (121) can be expressed as

$$f_1(\mathbf{r},\mathbf{v},t) = -\frac{e}{m}\int_{-\infty}^{t} \nabla\varphi_1(\mathbf{r}',t)\cdot\frac{\partial f_0}{\partial \mathbf{v}}\,dt'. \tag{122}$$

Expressing

$$(f_1,\varphi_1) = (f_{10},\varphi_{10})\exp i(k_z z + k_y y - \omega t), \tag{123}$$

and using (122), we can obtain the expression for $f_1$ in the form

$$\begin{aligned} f_1(v) \;&=\; -\frac{e}{m}\varphi_{10}\int_{-\infty}^{t} dt'\;[\exp\; i[k_z(z'-z) + k_y(y'-y) - \omega(t'-t)]]\times \\ &\times\, 2i[k_y v_y'\frac{\partial f_0}{\partial v_\perp'^2} + k_z v_z'\frac{\partial f_0}{\partial v_z'^2}]. \end{aligned} \tag{124}$$

The particle's trajectories can be expressed as

$$v_y' = v_\perp\cos(\omega_c\tau + \psi) \tag{125}$$

$$v_z' = v_z, \tag{126}$$

where $\tau = t' - t$, where $v_\perp$ is the perpendicular component of the velocity $\mathbf{v}$ and where $\omega_c = \frac{eB_0}{m}$ is the cyclotron frequency.

After the integration, we obtain

$$y' - y = \frac{v_\perp}{\omega_c}[\sin(\omega_c\tau + \psi) - \sin\psi]; \qquad z' - z = v_z\tau. \tag{127}$$

Using the identity

$$e^{i\gamma\sin\theta} = \sum_{n=-\infty}^{n=+\infty} J_n(\gamma)e^{in\theta}, \tag{128}$$

and using the Poisson equation, we obtain the dispersion relation in the form

$$k^2 + 8\pi^2\sum_{j=e,i}\frac{e_j^2}{m_j}\int dv_\perp^2 dv_z\sum_n\frac{J_n^2(\frac{k_\perp v_\perp}{\omega_c})}{\omega - k_z v_z - n\omega_c}[k_z v_z\frac{\partial f_0}{\partial v_z^2} + n\omega_c\frac{\partial f_0}{\partial v_\perp^2}] = 0. \tag{129}$$

As expected, the resonant condition has now more rich form

$$\omega - n\omega_c - k_z v_z = 0. \tag{130}$$

expressing the effect of the particle gyration by means of the cyclotron harmonics.

The solution of the dispersion relation is then given by the same procedure, as in the case without magnetic field.

## 3. THE QUASILINEAR THEORY

The quasilinear theory represents the simplest form of the theories, which describe the nonlinear interaction of waves with a plasma, and forms a part of the theory of weak turbulence. Originally, in the pioneering papers of Drummond and Pines [10] and of Vedenov, Velikhov and Sagdeev [11], the quasilinear theory was used to describe the saturation (or relaxation) of kinetic plasma instabilities. The mechanism seemed to be simple - the growing instability affects the velocity distribution function in a such way that the slope of the unstable part of the distribution function decreases, thus also decreasing the instability decrement.

The problem itself initiated a broad discussion of nonlinear and turbulent phenomena; the quasilinear theory is an important milestone in the development of these theories.

There exists yet another motivation for the quasilinear description of wave-plasma interaction - radiofrequency(RF) heating of plasma, and RF current drive. RF heating is used as an auxiliary source of power, suitable for heating plasma up to D-T ignition temperatutes. RF current drive, which can appear as a consequence of the absorption of RF waves in the plasma, can substitute the inductive tokamak current. RF plasma heating and RF current drive, are the typical effects of the interaction of RF fields with plasmas under kinetic regime.

Since the discussion of the selfconsistent evolution of instabilities and of RF plasma heating differ slightly in the definition of the problem, we shall describe both approaches separately.

For the investigation of the evolution of plasma turbulence within the frame of the quasilinear theory, the interaction of a warm electron beam, penetrating through a plasma, bump-on-tail instability, is the prototype example [12]. Generally, the quasilinear description of this form of the interaction represents a rather complicated problem. To describe the basic features of the quasilinear theory (QLT), the discussion of the 1D (one-dimensional) model of the beam-plasma interaction for the case of a homogeneous plasma without magnetic field is sufficient.

Let us therefore assume that the distribution function depends on space coordinate $x$, on velocity $v$ and on time $t$. The bump-on-tail instability distribution function

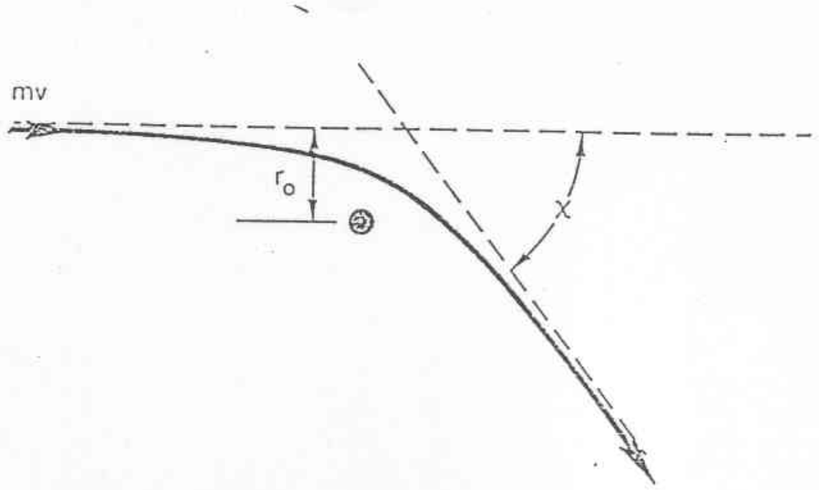
mv
$r_o$
$\chi$

Fig. 1

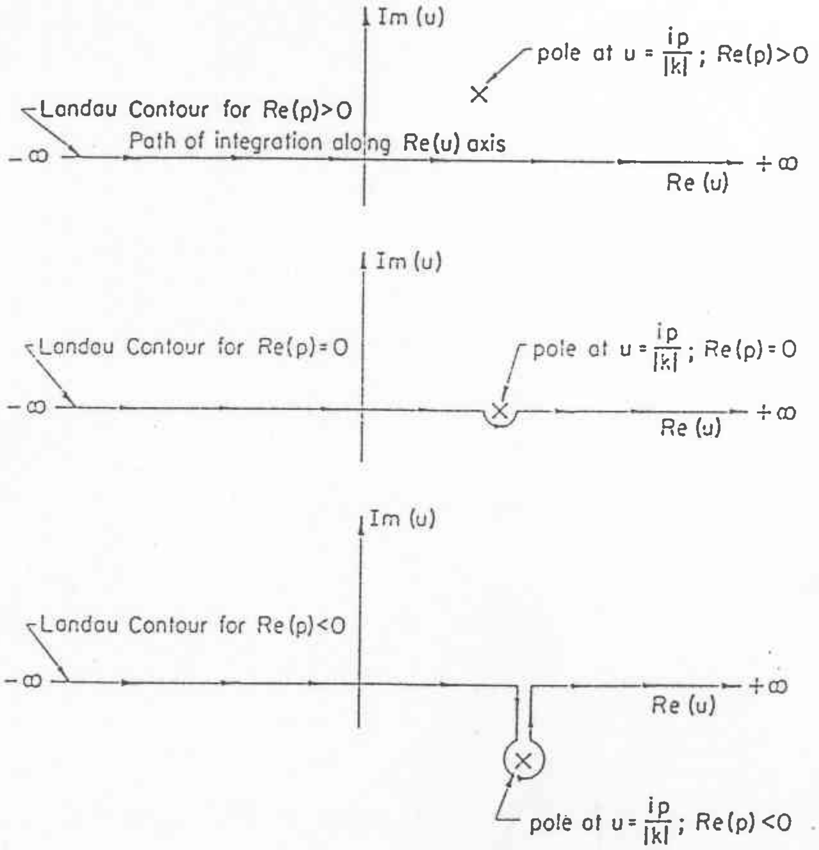
Im (u)
pole at $u = \frac{ip}{|k|}$; Re(p)>0
Landau Contour for Re(p)>0
Path of integration along Re(u) axis
$-\infty$
$+\infty$
Re (u)
Im (u)
Landau Contour for Re(p)=0
pole at $u = \frac{ip}{|k|}$; Re(p)=0
$-\infty$
$+\infty$
Re (u)
Im (u)
Landau Contour for Re(p)<0
$-\infty$
$+\infty$
Re (u)
pole at $u = \frac{ip}{|k|}$; Re(p)<0

Fig. 2

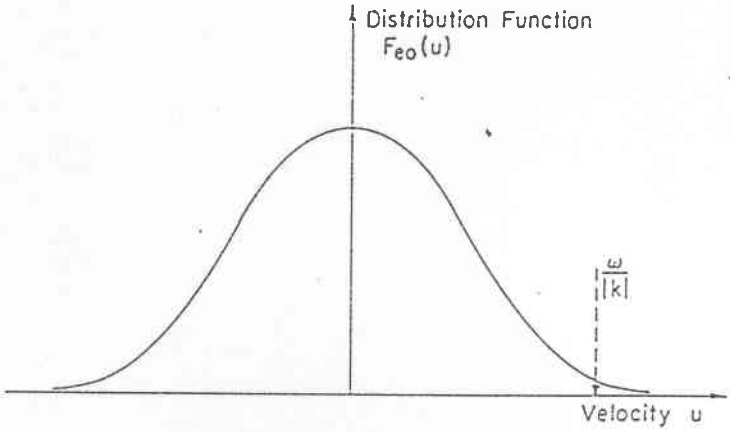
Distribution Function
$F_{eo}(u)$
$\frac{\omega}{|k|}$
Velocity u

Fig. 3

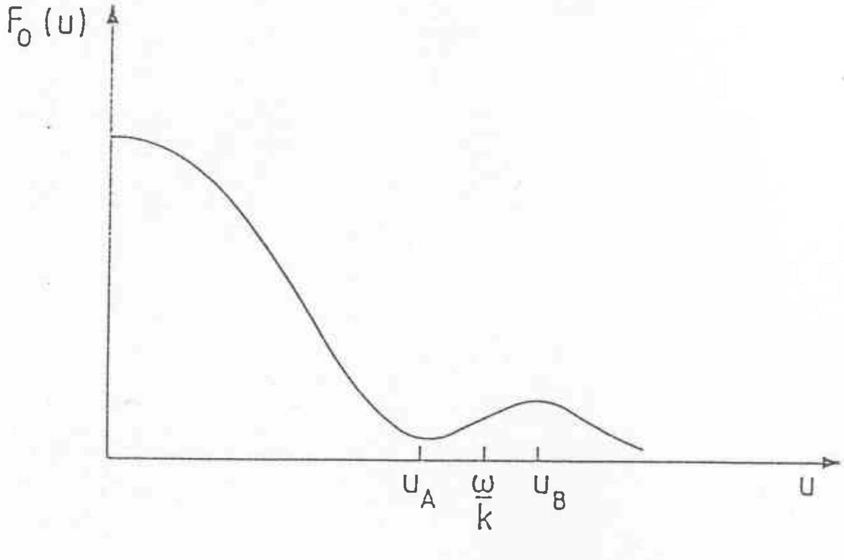
$F_0(u)$
$u_A$
$\frac{\omega}{k}$
$u_B$
u

Fig. 4

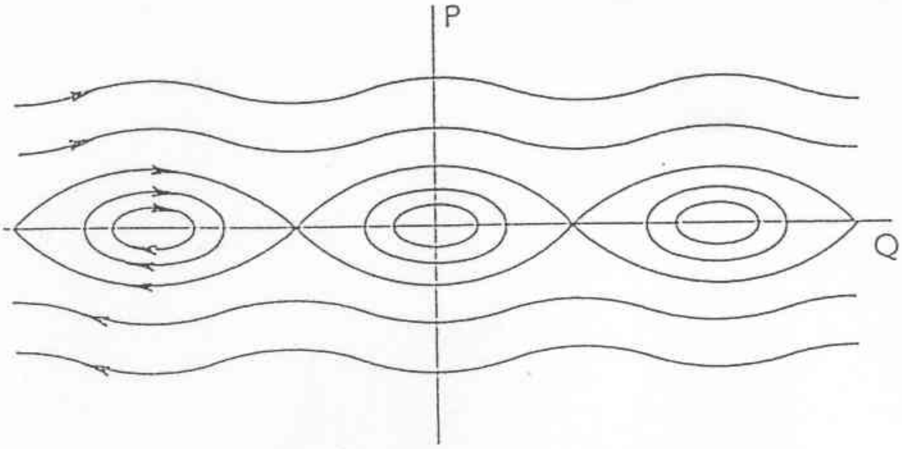


Fig. 5

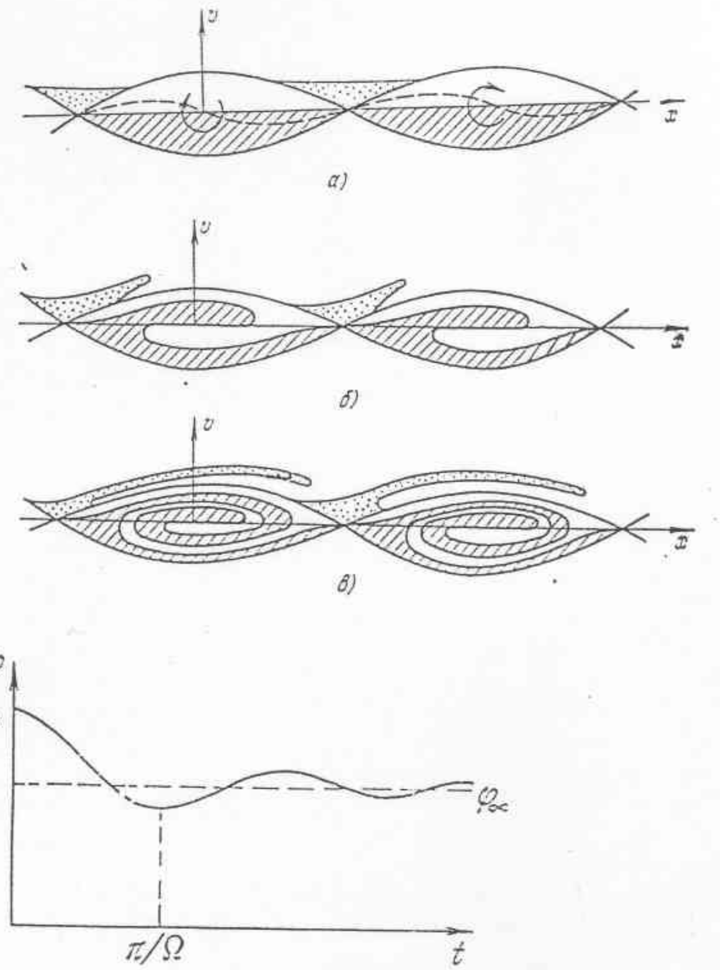


Fig. 6.