Lecture Notes NAST043 (Planet formation)

RNDr. Ondřej Chrenko, Ph.D.

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1 Dust dynamics and evolution

1.1 Radial and orbital velocity of small solids

We study the motion of a test particle, representing a dust grain (or a pebble), orbiting a star ($M_\star$) while embedded in a gaseous protoplanetary disk ([1]). The equation of motion thus contains the central gravitational acceleration and the aerodynamic drag acceleration arising due to the relative motion between the grain and the gas.

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = -\frac{GM_\star}{r^3}\boldsymbol{r} + \boldsymbol{a_\mathrm{D}} = -\frac{GM_\star}{r^3}\boldsymbol{r} + a_\mathrm{D}\frac{\boldsymbol{v_\mathrm{rel}}}{v_\mathrm{rel}}\,, \tag{1}$$

We used the fact that the drag acceleration[1] will be directed along the relative velocity vector defined as $\boldsymbol{v_\mathrm{rel}} = \boldsymbol{v} - \boldsymbol{u}$ (gas lagging behind the grain will slow it down).

Since the gas flow is naturally decomposed into a (fast) orbital motion and (slow) viscously-driven radial movement, we transform into 2D polar coordinates. The velocity vector becomes $\boldsymbol{u} = u_r\hat{\boldsymbol{e}}_{\boldsymbol{r}} + u_\phi\hat{\boldsymbol{e}}_{\boldsymbol{\phi}}$ and to differentiate, one has to realize that the basis vectors are not time-independent. Using $\hat{\boldsymbol{e}}_{\boldsymbol{r}} = (\cos\phi, \sin\phi)$ and $\hat{\boldsymbol{e}}_{\boldsymbol{\phi}} = (-\sin\phi, \cos\phi)$, we obtain $\dot{\hat{\boldsymbol{e}}}_{\boldsymbol{r}} = (-\dot\phi\sin\phi, \dot\phi\cos\phi) = \dot\phi\hat{\boldsymbol{e}}_{\boldsymbol{\phi}}$ and $\dot{\hat{\boldsymbol{e}}}_{\boldsymbol{\phi}} = (-\dot\phi\cos\phi, -\dot\phi\sin\phi) = -\dot\phi\hat{\boldsymbol{e}}_{\boldsymbol{r}}$. By definition, we have $\dot\phi = u_\phi/r$ and then

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \dot{u_r}\hat{\boldsymbol{e}}_{\boldsymbol{r}} + u_r\left(\frac{u_\phi}{r}\hat{\boldsymbol{e}}_{\boldsymbol{\phi}}\right) + \dot{u_\phi}\hat{\boldsymbol{e}}_{\boldsymbol{\phi}} + u_\phi\left(-\frac{u_\phi}{r}\right)\hat{\boldsymbol{e}}_{\boldsymbol{r}} = \left(\dot{u_r} - \frac{u_\phi^2}{r}\right)\hat{\boldsymbol{e}}_{\boldsymbol{r}} + \left(\dot{u_\phi} + \frac{u_r u_\phi}{r}\right)\hat{\boldsymbol{e}}_{\boldsymbol{\phi}}\,. \tag{2}$$

The $\propto u_r^2$ term is the centrifugal term and the $\propto u_r u_\phi$ is the curvature term.

Let us merge Eqs. (1) and (2) component-wise and move from the Lagrangian to the Eulerian framework. We will do so by using an axisymmetric assumption (no ∂_ϕ terms) and applying the chain rule along the particle trajectory, so for $u_r(t, r(t))$ we have $\dot{u_r} = \partial_t u_r + \dot r\partial_r u_r = \partial_t u_r + u_r\partial_r u_r$, and similarly for u_ϕ. We obtain

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} - \frac{u_\phi^2}{r} = -\frac{GM_\star}{r^2} - \frac{a_\mathrm{D}}{v_\mathrm{rel}}\left(u_r - v_r\right)\,, \tag{3a}$$

$$\frac{\partial u_\phi}{\partial t} + u_r\frac{\partial u_\phi}{\partial r} + \frac{u_r u_\phi}{r} = -\frac{a_\mathrm{D}}{v_\mathrm{rel}}\left(u_\phi - v_\phi\right)\,. \tag{3b}$$

Assuming a steady-state situation, we set $\partial_t = 0$. Furthermore, we assume that the deviations in the grain's velocity caused by the drag remain highly sub-Keplerian. One can then decompose $u_\phi = v_\mathrm{K} + u'_\phi$ with the ordering $|u'_\phi| \lesssim \delta \ll v_\mathrm{K}$ and assume the same ordering for the total radial velocity $|u_r| \lesssim \delta \ll v_\mathrm{K}$.

In Eq. (3a), the first two terms drop out and after multiplying by $(-1/r)$, we write

$$v_\mathrm{K}^2 + 2v_\mathrm{K}u'_\phi + \mathcal{O}\left(\delta^2\right) = v_\mathrm{K}^2 + \frac{a_\mathrm{D}}{v_\mathrm{rel}}r\left(u_r - v_r\right) \tag{4}$$

which is equivalent to

$$2\Omega_\mathrm{K}u'_\phi = \frac{a_\mathrm{D}}{v_\mathrm{rel}}\left(u_r - v_r\right)\,. \tag{5}$$

[1] Note that $\boldsymbol{a_\mathrm{D}}$ is often denoted as $\boldsymbol{F_\mathrm{D}}$ in literature but here we use a notation exemplifying that the term represents an acceleration rather than a force.

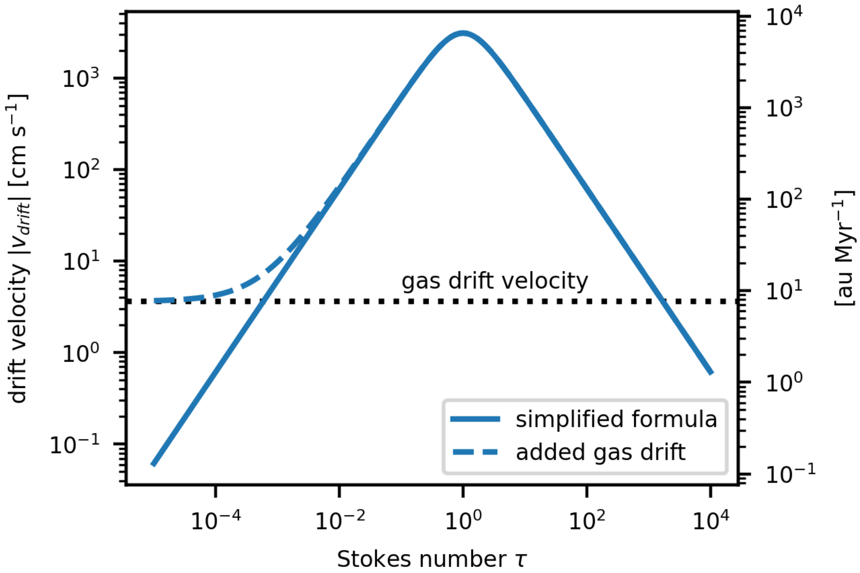

Figure 1: Magnitude of the inward-directed radial drift velocity of dust grains as a function of their Stokes number. The solid curve includes only the first term of Eq. (9), while the dashed curve shows the full expression, including the contribution of the radial gas motion (driven by a turbulent viscosity in this example). The disk model is based on simple power-law profiles for the surface density and temperature, following the setup from [2]. In this model, the drift velocity does not depend on the orbital distance within the disk.

Using the definition of the Stokes number $\tau = t_\mathrm{s}\Omega_\mathrm{K} = v_\mathrm{rel}\Omega_\mathrm{K}/a_\mathrm{D}$, where t_s is the stopping time, we obtain

$$u'_\phi = \frac{1}{2\tau}\left(u_r - v_r\right). \tag{6}$$

The left-hand side of Eq. (3b) retains two terms and the whole equation can be rewritten as

$$u_r\frac{\partial v_\mathrm{K}}{\partial r} + \frac{u_r v_\mathrm{K}}{r} + \mathcal{O}\left(\delta^2\right) = u_r\sqrt{GM_\star}\left(-\frac{1}{2}r^{-3/2}\right) + u_r\Omega_\mathrm{K} = \frac{1}{2}u_r\Omega_\mathrm{K} = -\frac{a_\mathrm{D}}{v_\mathrm{rel}}\left(v_\mathrm{K} + u'_\phi - v_\phi\right). \tag{7}$$

For the right-hand side, we recall $v_\phi = (1-\eta)v_\mathrm{K}$ and we use the definition of the Stokes number again:

$$u_r = -\frac{2}{\tau}(u'_\phi + \eta v_\mathrm{K}). \tag{8}$$

Next, we plug Eq. (6) to Eq. (8) and after trivial rearrangements, we arrive to

$$\boxed{u_r = -\frac{2\tau}{1+\tau^2}\left(\eta v_\mathrm{K} - \frac{1}{2\tau}v_r\right).} \tag{9}$$

Then, using Eq. (9) in Eq. (6), the full azimuthal velocity $u_\phi = v_\mathrm{K} + u'_\phi$ becomes[2]

$$\boxed{u_\phi = v_\mathrm{K} - \frac{1}{1+\tau^2}\left(\eta v_\mathrm{K} + \frac{\tau}{2}v_r\right).} \tag{10}$$

[2] In literature, Eq. (10) can sometimes be encountered with an incorrect negative last term [3, 4]. See e.g. [5] for a correct expresion.

From Eq. (9), it is useful to isolate the term which is 'always present', even when there is no radial movement in the gas ($v_r = 0$). This so-called radial drift velocity reads

$$v_{\mathrm{drift}} = -\frac{2\tau}{1+\tau^2}\eta v_{\mathrm{K}} \overset{\tau \ll 1}{\simeq} -2\tau\eta v_{\mathrm{K}}\,. \tag{11}$$

The negative sign tells us that the velocity is directed towards smaller radii in the stellarcentric frame for the typical situation when $\eta > 0$. In other words, the drift motion forces small solids to spiral towards the disk centre. The right-hand-side form of Eq. (11) provides a useful simple expression for the smallest dust grains whose coupling to the gas disk is strong and timescale of momentum exchange is short.

To find limiting values that v_{drift} can attain, we denote $f(\tau) = 2\tau/(1+\tau^2)$ and calculate

$$f'(\tau) = \frac{2}{1+\tau^2} + 2\tau\frac{-1}{(1+\tau^2)^2}2\tau = \frac{2(1+\tau^2)-4\tau^2}{(1+\tau^2)^2} = 2\frac{(1-\tau)(1+\tau)}{(1+\tau^2)^2}\,, \tag{12}$$

which immediately gives as a critical point for $\tau = 1$ (τ is always positive by definition). The sign of $f'(\tau)$ tells us that $f(\tau)$ is increasing (decreasing) for $\tau < 1$ (> 1), therefore $v_{\mathrm{drift}} = -\eta v_{\mathrm{K}}$ is the maximum drift velocity and is expected for particles with $\tau = 1$.

An example dependence of the drift velocity on the grain Stokes number is shown in Fig. 1. It reveals that the radial drift is so efficient as to deplete a 100-au-sized protoplanetary disk of all grains with $\mathrm{St} \gtrsim 10^{-2}$ within its lifetime (~few Myr). This is known as the drift barrier for dust growth: the bigger the dust grain gets by coagulation, the more efficiently it spirals towards the central star.

It is important to note that although protoplanetary disks tend to have an inward-directed global pressure gradient, there are processes that can locally facilitate inversions of ∇P. Since Eq. (9) dictates that dust grains always drift towards a region of higher pressure, a pressure gradient inversion can revert an inward drift to an outward drift. In other words, if the gas disk becomes locally super-Keplerian, particles start feeling tailwind rather than headwind. Provided that the global pressure gradient is directed towards the star, a local ∇P inversion creates a zone of convergent dust drift, accumulating dust grains arriving from larger radii and creating a dust trap. These dust trapping regions are commonly referred to as pressure bumps and are thought to lie at the origin of dust-abundant rings discovered in many protoplanetary disks by ALMA. Pressure bumps can provide a possible solution to the drift-driven depletion problem and can also work as suitable sites for the growth of small solids into larger bodies.

Finally, it is important to remember that the aerodynamic drag is a two-way force and to conserve momentum, the global-scale inward drift of dust has to translate to an outward drift of gas (see [6, 7]). This effect is typically negligible due to the canonically low dust-to-gas ratio. But whenever the latter gets elevated, the back-reaction of dust can play a crucial dynamical role.

References

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