Fig. 1   Surface pressure evolution at gridpoint 0°E, 52°N for forecast started from uninitialised (full) and initialised (dashed) analysis for 12 GMT 6 September 1982.
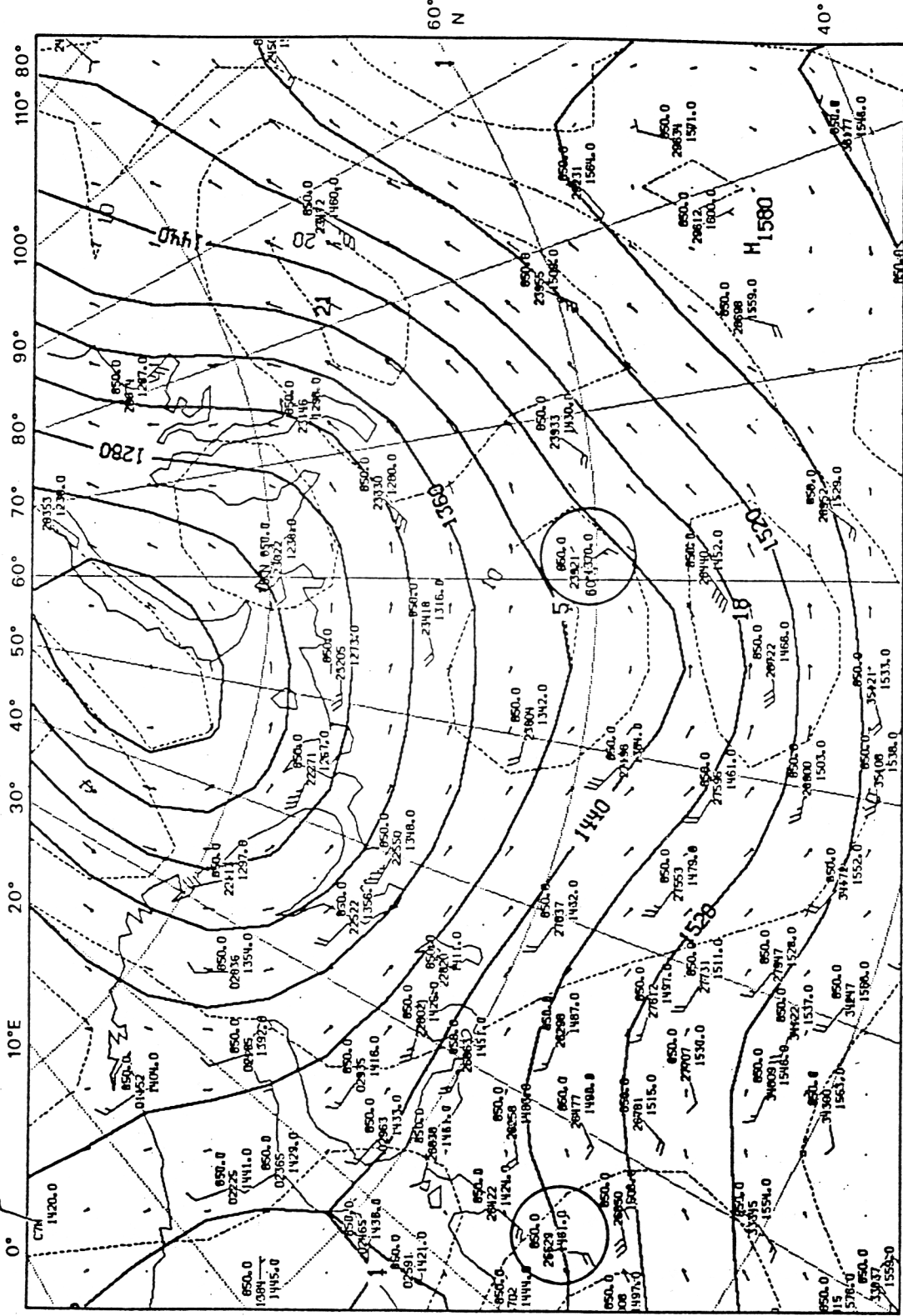
Fig. 2   850 mb observations and first guess height (full) and wind field.
Contour interval for isotachs (dashed) is 10 m/s.   Circles indicate
differently treated observations.

2. INITIALISATION OF A SIMPLE SHALLOW WATER MODEL

In this Section the bi-periodic shallow water model on an f-plane is used in order to introduce the concepts of non-linear normal mode initialisation. This type of model allows a straightforward analytical treatment and is at the same time still relevant for more complex models.

The model equations are:

$$\frac{\partial u}{\partial t} - f_o v + \frac{\partial \phi'}{\partial x} = - u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} + F_x = N_u \tag{2.1}$$

$$\frac{\partial v}{\partial t} + f_o u + \frac{\partial \phi'}{\partial y} = - u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} + F_y = N_v \tag{2.2}$$

$$\frac{\partial \phi'}{\partial t} + \bar{\phi} (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = - \phi' (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) - u\frac{\partial \phi'}{\partial x} - v\frac{\partial \phi'}{\partial y} + Q = N_\phi \tag{2.3}$$

These equations describe the motions of an incompressible fluid with a free surface. They can be derived from the primitive equations by assuming that the density ρ depends only upon the vertical coordinate and that the initial wind is constant with height. ϕ, the geopotential of the free surface, has been split into a constant part $\bar{\phi}$ and a deviation ϕ'. The velocity components in the x and y directions are u and v, and f_o is the (constant) Coriolis parameter. Frictional effects are given by F_x and F_y, and Q is a heating term. The non-linear terms on the right hand side of (2.1), (2.2) and (2.3) have been combined and are represented by N_u, N_v and N_ϕ.

2.1 DERIVATION OF THE NORMAL MODES

Normal modes are the free motions of a system which is capable of vibrating. A typical example is a guitar string; once it has been excited it starts to vibrate in a way characteristic of that particular string. These vibrations

are called "own" or "eigen" vibrations, or in mathematical terms "normal modes".

In the following Section the normal modes of (2.1) - (2.3) are derived and discussed.

As a bi-periodic domain has been assumed, one may use double Fourier series to expand the dependent variable u, v, ϕ' and (symbolically) the non-linear terms N.

$$
\begin{bmatrix} u \\ v \\ \phi' \\ N \end{bmatrix} = \sum_{n=-M}^{M} \sum_{m=-M}^{M} \begin{bmatrix} u_n^m \\ iv_n^m \\ \sqrt{\bar{\phi}}\ \phi_n^m \\ N_n^m \end{bmatrix} e^{i(\frac{2\pi mx}{L_x} + \frac{2\pi my}{L_y})} \tag{2.4}
$$

M is the maximum zonal and meridional number of waves, with L_x and L_y the zonal and meridional extent of the domain. The factor $i = \sqrt{-1}$ in front of the Fourier coefficient v_n^m results in a phase shift of a quarter of a wavelength; also the factor $\sqrt{\bar{\phi}}$ in front ϕ_n^m is required later on for consistency of dimensions. Both factors simplify the subsequent analyses.

Inserting (2.4) into (2.1) - (2.3) yields

$$
\frac{\partial}{\partial t} u_n^m = f_o iv_n^m - ik\sqrt{\bar{\phi}}\ \phi_n^m + N_{un}^m \tag{2.5}
$$

$$
\frac{\partial}{\partial t} iv_n^m = -f_o u_n^m - il\sqrt{\bar{\phi}}\ \phi_n^m + N_{vn}^m \tag{2.6}
$$

$$
\frac{\partial}{\partial t} \sqrt{\bar{\phi}}\ \phi_n^m = -\bar{\phi}\ iku_n^m - \bar{\phi}i^2 lv_n^m + N_{\phi N}^m \tag{2.7}
$$

Here, $k = 2\pi m/L_x$ and $l = 2\pi m/L_y$ are the zonal and meridional wavenumbers. Now define a vector $\underline{X}_n^m$ as

$$\underline{X}_n^m = \begin{bmatrix} u_n^m \\ v_n^m \\ \phi_n^m \end{bmatrix} \qquad (2.8)$$

Multiplying (2.6) by $-i$ and dividing (2.7) by $\sqrt{\bar{\phi}}$ gives a system of equations which may be written in matrix form as

$$\frac{\partial}{\partial t} \underline{X}_n^m = i\underline{\underline{A}}_n^m \underline{X}_n^m + \underline{N}_n^m \qquad (2.9)$$

where $\underline{N}_n^m$ is the vector of the non-linear terms

$$\underline{N}_n^m = \begin{bmatrix} N_{un}^m \\ -iN_{vn}^m \\ (\bar{\phi})^{-\frac{1}{2}} N_{\phi n}^m \end{bmatrix} \qquad (2.10)$$

In the following, the reference to the zonal and meridional components m and n will be dropped. However, one has to keep in mind that there are different matrices and vectors for every m and n. The matrix $\underline{\underline{A}}$ is given by:

$$\underline{\underline{A}} = \begin{bmatrix} 0 & f & -k\sqrt{\bar{\phi}} \\ f_0 & 0 & il\sqrt{\bar{\phi}} \\ -k\sqrt{\bar{\phi}} & -il\sqrt{\bar{\phi}} & 0 \end{bmatrix} \qquad (2.11)$$

It is now possible to find the normal modes (eigensolutions of the linearised version ($\underline{N}_n^m = 0$) of (2.9)

First one has to find to the eigenvalues of $\underline{\underline{A}}$. As $\underline{\underline{A}}$ is a Hermitian matrix (rows identical to complex conjugate columns) only real eigenvalues are expected λ_j. They can easily be computed.

$$\lambda_1 = 0 \tag{2.12}$$

$$\lambda_{2,3} = \pm[f_o^2 + \bar{\phi}(k^2+\ell^2)]^{1/2} = \pm\sigma \quad \left(\text{frequency}\right) \tag{2.13}$$

$\lambda_{2,3}$ are the well known frequencies for inertia-gravity waves; λ_1 is the frequency of the Rossby waves which is stationary in this case because a constant f_o has been assumed. The modulus of the phase velocity for the inertia gravity waves is given by

$\cdot \frac{\sigma}{k} = \frac{\text{frequency}}{\text{wavenumber}}$

$$|c| = \frac{\sigma}{\sqrt{k^2+\ell^2}} = [\bar{\phi} + \frac{f_o^2}{k^2 + \ell^2}]^{1/2}. \tag{2.14}$$

The phase velocity for inertia-gravity waves depends upon the wavenumbers k and l (i.e. the waves are dispersive). Thus the waves can distribute an initially locally confined quantity over a larger area.

Now the eigenvectors (normal modes) of $\underline{\underline{A}}$ are derived by solving

$$\underline{\underline{A}}\underline{v}_j = \lambda_j \underline{v}_j \quad j = 1,2,3 \tag{2.15}$$

for the eigenvectors $\underline{v}_j$ (these are column matrices).

Using λ_1 one may find the corresponding eigenvector $\underline{v}_1$ (now denoted by $\underline{R}$),

$A\vec{x} = \lambda\vec{x}$

$$\underline{R} = \begin{bmatrix} -i\ell\sqrt{\bar{\phi}}/f_o \\ k\sqrt{\bar{\phi}}/f_o \\ 1 \end{bmatrix} \tag{2.16}$$

This is the so called Rossby mode. It is immediately evident that it establishes, at least in this model, a geostrophic coupling between the geopotential (here arbitrarily scaled to 1) and the wind components.

For the two gravity modes (j=2,3) the eigenvectors may be written as

$$\underline{G}_{E,W} = \begin{bmatrix} if_o l \mp \sigma k \\ -f_o k \pm i\sigma l \\ \sqrt{\bar{\phi}}(k^2+l^2) \end{bmatrix} \qquad (2.17)$$

The upper sign in front of σ gives the eastward travelling gravity mode and the lower sign the westward one. As for the Rossby mode, the gravity modes are also characterised by a specific scale dependent coupling between the mass and wind fields.

Problem 1

Make a schematic sketch of

    (i)   a Rossby mode for $m=n=1$

    (ii) a gravity mode for $m=o$, $n=1$

2.2   PROJECTION ON TO NORMAL MODES

An important property of normal modes is their orthogonality; it can also be shown that they form a complete set. Therefore any vector $\underline{X}$ may be expanded in terms of a series of normal modes in the same way as a field can be expanded as a Fourier series. For this purpose a modal matrix $\underline{\underline{E}}$ is defined, which has the 3 eigenvectors as its columns

$$\underline{\underline{E}} = \frac{1}{\sigma} \begin{bmatrix} -il\sqrt{\bar{\phi}} & \dfrac{if_o l - \sigma k}{\sqrt{2(k^2+l^2)}} & \dfrac{if_o l + \sigma k}{\sqrt{2(k^2+l^2)}} \\[3mm] k\sqrt{\bar{\phi}} & \dfrac{-f_o k + i\sigma l}{\sqrt{2(k^2+l^2)}} & \dfrac{-f_o k - i\sigma l}{\sqrt{2(k^2+l^2)}} \\[3mm] f_o & \dfrac{\sqrt{\bar{\phi}}(k^2+l^2)}{\sqrt{2(k^2+l^2)}} & \dfrac{\sqrt{\bar{\phi}}(k^2+l^2)}{\sqrt{2(k^2+l^2)}} \end{bmatrix} \qquad (2.18)$$

In (2.18) the modes have been normalised to length unity. To project an arbitrary vector $\underline{X}$ onto the normal modes, one writes

$$\underline{X} = \underline{\underline{E}}\underline{y} \qquad (2.19)$$

$\underline{X}$ is the known vector to be protected and $\underline{y}$ holds its normal mode components. To get $\underline{y}$, multiply (2.19) from the left by $\underline{\underline{E}}^{-1}$. This yields

$$\underline{y} = \underline{\underline{E}}^{-1}\underline{X} \qquad (2.20)$$

As $\underline{\underline{E}}$ is orthogonal, $\underline{\underline{E}}^{-1}$ can simply be computed by making a conjugate transposition. The component form of (2.20) may then be written as

$$y_R = \frac{1}{\sigma} \ [+il\sqrt{\bar{\phi}}\ u + k\sqrt{\bar{\phi}}\ v + f_o\phi] \qquad (2.21)$$

$$y_{GE} = \frac{1}{\sigma\sqrt{2(k^2+l^2)}} \ [-(if_o l + \sigma k)u - (f_o k + i\sigma l)v + \sqrt{\bar{\phi}}(k^2+l^2)\phi] \qquad (2.22)$$

$$y_{GW} = \frac{1}{\sigma\sqrt{2(k^2+l^2)}} \ [-(if_o l - \sigma k)u - (f_o k - i\sigma l)v + \sqrt{\bar{\phi}}(k^2+l^2)\phi] \qquad (2.23)$$

For given values of u, v and ϕ (for example analysed values) (2.21) - (2.23) shows how these fields project onto Rossby (2.21) and gravity modes (2.22,2.23). If the mass and wind fields are in geostrophic balance, they do not excite gravity waves, i.e. $y_{GE} = y_{GW} = 0$. On the other hand, a geopotential amplitude ϕ alone (without wind amplitudes, u,v) projects onto both Rossby and gravity modes. All the projections are scale dependent; for instance small scale wind fields project more on Rossby modes than large scale ones. In the atmosphere, gravity mode amplitudes are usually much smaller than Rossby mode amplitudes.

Problem 2

    (i)   Show that the normal modes are othogonal.

    (ii) Show that geostrophic winds do not project on gravity
         modes for this model.

(iii) Derive mass and wind fields which do not project on
Rossby waves.

2.3 MODEL EQUATIONS IN NORMAL MODE FORM

Inserting (2.19) (and a corresponding transformation for the vector of non-
linear terms $\underline{N}$) into (2.9) yields:

$$\frac{\partial}{\partial t} \underline{E}\underline{y} = i\underline{A}\underline{E}\underline{y} + \underline{E}\underline{q} \tag{2.24}$$

$\underline{q}$ is the vector of normal mode amplitudes of $\underline{N}$.

Multiplication of (2.24) by $\underline{\underline{E}}^{-1}$ from the left results in:

$$\frac{\partial \underline{y}}{\partial t} = i\underline{\underline{E}}^{-1} \underline{\underline{A}}\underline{\underline{E}}\underline{y} + \underline{q} \tag{2.25}$$

The similarity transform $\underline{\underline{E}}^{-1} \underline{\underline{A}} \underline{\underline{E}}$ reduced $\underline{\underline{A}}$ to a diagonal matrix $\underline{\underline{D}}$ which holds

the eigenvalues λ_j as its diagonal elements and zero entries elsewhere.

Therefore, a decoupled system is obtained which can be written in component

form.

$$\frac{dy_R}{dt} = i\lambda_1 y_R + q_R \tag{2.26}$$

$$\frac{dy_{GE}}{dt} = i\lambda_2 y_{GE} + q_{GE} \tag{2.27}$$

$$\frac{dy_{GW}}{dt} = i\lambda_3 y_{GW} + q_{GW} \tag{2.28}$$

These are the model equations in normal mode form. They form a set of

decoupled, ordinary differential equations equivalent to the original system

of coupled partial differential equations (2.1 to 2.3). Neglecting the non-

linear terms, (2.26) – (2.28) can be integrated to give

$$Y_R(t) = Y_R(t=o) , \tag{2.29}$$

$$Y_{GE}(t) = Y_{GE}(t=o)e^{i\sigma t} , \tag{2.30}$$

$$Y_{GW}(t) = y_{GW}(t{=}0)e^{-i\sigma t} . \qquad (2.31)$$

(2.29) - (2.31) are the analytical solutions of the linearised version of the set of Equations (2.1)-(2.3). Once an analysis has provided the initial values of u, v and ϕ, (2.21) - (2.23) can be used to get $y_R(t{=}0)$, $y_{GE}(t{=}0)$ and $y_{GW}(t{=}0)$ and then (2.29) - (2.31) used to compute the value of the normal mode coefficients at a subsequent time t. Using (2.19), (2.8) and the inverse of (2.4) then allows the computation of the mass and wind fields in physical space. From (2.29) it can be seen that the Rossby mode coefficient y_R remains constant in time. However the gravity mode coefficients will oscillate with their (high) frequency σ unless their initial amplitudes $y_{GE}(t{=}0)$ and $y_{GW}(t{=}0)$ are zero.

At this point it is worth recalling that inertia-gravity waves are dispersive. Suppose that there is a zero wind field and a localised mass field disturbance at only a single point. This state will excite both types of waves within a wide spectrum of horizontal scales. Each inertia-gravity wave will travel away from the disturbance with its own characteristic phase speed thus leading to a broadening of the shape of the gravity mode part of the initial state. This will finally result in a state where only the Rossby wave components remain in the region of the initial disturbance. The transition process is known as geostrophic adjustment.

2.4   LINEAR NORMAL MODE INITIALISATION

In order to suppress unwanted oscillations in the linear model, it is clear from (2.30) and (2.31) that one must set $y_{GE}(t{=}0)$ and $y_{GW}(t{=}0)$ for the gravity modes to zero. Using (2.22) and (2.23) one obtains

$$\frac{1}{\sigma\sqrt{2(k^2+1^2)}} [(if_o1 - \sigma k)u(t=o) - (f_ok - i\sigma l)v(t=o) + \sqrt{\bar{\phi}}(k^2+1^2)\phi(t=o)] = 0 \qquad (2.32)$$

$$\frac{1}{\sigma\sqrt{2(k^2+1^2)}} [(if_o1 + \sigma k)u(t=o) - (f_ok + i\sigma l)v(t=o) + \sqrt{\bar{\phi}}(k^2+1^2)\phi(t=o)] = 0 \qquad (2.33)$$

There are various ways of satisfying (2.32) and (2.33). Usually one requires

obvykle se pp., že Rossby mód není inicializ. změněn.

that the Rossby mode component y_R is not changed by the initialisation. This

Toho se dosáhne odečtením gravitačních vln z počátečního pole

can be achieved simply by subtracting the gravity wave components from the

initial field.

$$u_{LI} = u(t=o) - \frac{1}{\sigma\sqrt{2(k^2+1^2)}} [(if_o1-\sigma k)y_{GE}(t=o)+(if_o1+\sigma k)y_{GW}(t=o)] \qquad (2.34)$$

$$v_{LI} = v(t=o) - \frac{1}{\sigma\sqrt{2(k^2+1^2)}} [(-f_ok+i\sigma l)y_{GE}(t=o)-(f_ok+i\sigma l)y_{GW}(t=o)] \qquad (2.35)$$

$$\phi_{LI} = \phi(t=o) - \frac{\sqrt{\bar{\phi}}(k^2+1^2)}{\sigma\sqrt{2(k^2+1^2)}} [y_{GE}(t=o) + y_{GW}(t=o)] \qquad (2.36)$$

By modifying the analysed initial state according to (2.34) - (2.36) it is

místně

ensured that the gravity wave amplitudes vanish initially and as a consequence

of (2.30) and (2.31) - for all time if integration is performed with the

pro který čas integrace je provedena s

linearised version of (2.1)-(2.3).

linearizovanou verzí (2.1)-(2.3)

It is interesting to look at the relative changes to the mass ($\Delta\phi_{LI}$) and wind

fields (Δu_{LI}, Δv_{LI}) due to linear normal mode initialisation. If for

convenience the gravity mode components y_{GE} and y_{GW} are treated as identical,

považovány za identické

one obtains *obdržíme*

$$\frac{\Delta\phi_{LI}}{\Delta u_{LI}} = \frac{-i\sqrt{\bar{\phi}}(k^2+1^2)}{f_o l} \qquad (2.37)$$

$$\frac{\Delta\phi_{LI}}{\Delta v_{LI}} = \frac{-i\sqrt{\bar{\phi}}(k^2+1^2)}{f_o k} \qquad (2.38)$$

From (2.37), (2.38) it is seen that the wind field changes decrease with increasing value of the Coriolis parameter. In other words, when approaching the tropics, the wind field becomes more dominant. Furthermore, the mass field changes increase with increasing mean depth ϕ of the fluid. Finally, the larger the horizontal scale, the more the wind field changes when compared to the mass field. All these results are consistent with geostrophic adjustment theory.

Problem 3

(i) Show that the initialised fields defined by (2.34)-(2.36) yield vanishing gravity mode projections.

(ii) Show that (2.34)-(2.36) do not change the Rossby wave projections.

(iii) Using (2.32) and (2.33) derive equations which do not change the mass field. Do these relations change the Rossby mode projections?

2.5 NON-LINEAR NORMAL MODE INITIALISATION

The relations (2.34)-(2.36) guarantee that, for a linear model, the gravity modes will vanish initially and stay zero for all time during the integration. However, when the same initial conditions are used to integrate a non-linear model, (2.26) - (2.28) show that the non-linear terms q_{GE} and q_{GW} will give rise to time tendencies, although Y_{GE} and Y_{GW} are initially zero. In other words, the gravity wave modes will not remain zero for a non-linear model. Therefore, one must use a different initialisation procedure. Machenhauer (1977) proposed that the initial tendencies should be set to zero. From (2.26)-(2.28) one then obtains

$$Y_{GE} = \frac{iq_{GE}}{\sigma} \tag{2.39}$$