1

Keplerovské elementy nejsou vhodné pro popis případů, kdy excentricita a/nebo sklon jsou malé veličiny. V tomto ohledu případě se zavádějí tzv. nesingulární elementy a my nyní potřebujeme formulovat Lagrangeovy pohybové rovnice v těchto veličinách.

V prvním kroku přejdeme k novým proměnným ϖ a λ

$$(\Omega, \omega, l) \longmapsto (\Omega, \varpi, \lambda): \qquad \boxed{\begin{aligned} \Omega &= \Omega \\ \varpi &= \Omega + \omega \\ \lambda &= \Omega + \omega + l \end{aligned}}$$

pak

$$\frac{\partial}{\partial \Omega} = \frac{\partial \Omega}{\partial \Omega}\frac{\partial}{\partial \Omega} + \frac{\partial \varpi}{\partial \Omega}\frac{\partial}{\partial \varpi} + \frac{\partial \lambda}{\partial \Omega}\frac{\partial}{\partial \lambda} = \frac{\partial}{\partial \Omega} + \frac{\partial}{\partial \varpi} + \frac{\partial}{\partial \lambda}$$

$$\frac{\partial}{\partial \omega} = \frac{\partial \varpi}{\partial \omega}\frac{\partial}{\partial \varpi} + \frac{\partial \lambda}{\partial \omega}\frac{\partial}{\partial \lambda} = \frac{\partial}{\partial \varpi} + \frac{\partial}{\partial \lambda}$$

$$\frac{\partial}{\partial l} = \frac{\partial \lambda}{\partial l}\frac{\partial}{\partial \lambda} = \frac{\partial}{\partial \lambda}$$

Pozn. bylo by lepší zavést jiné ozn. Ω'...

Lagrangeovy rovnice v těchto proměnných tedy vypadají takto

$$\dot{a} = -\frac{2}{na}\frac{\partial R}{\partial \lambda} \qquad\qquad 1-\eta = \frac{e^2}{1+\eta}$$

$$\dot{e} = \frac{\eta}{na^2 e}\left[\frac{\partial R}{\partial \varpi} + \frac{\partial R}{\partial \lambda} - \eta\frac{\partial R}{\partial \lambda}\right] = \frac{\eta}{na^2 e}\left[\frac{\partial R}{\partial \varpi} + (1-\eta)\frac{\partial R}{\partial \lambda}\right]$$

$$= \frac{\eta}{na^2 e}\left[\frac{\partial R}{\partial \varpi} + \frac{e^2}{1+\eta}\frac{\partial R}{\partial \lambda}\right]$$

$$\dot{i} = \frac{1}{na^2\eta \sin i}\left[\frac{\partial R}{\partial \Omega} + \frac{\partial R}{\partial \tilde{\omega}} + \frac{\partial R}{\partial \lambda} - \cos i \frac{\partial R}{\partial \tilde{\omega}} - \cos i \frac{\partial R}{\partial \lambda}\right]$$

$$= \frac{1}{na^2\eta \sin i}\left[\frac{\partial R}{\partial \Omega} + (1-\cos i)\left(\frac{\partial R}{\partial \tilde{\omega}} + \frac{\partial R}{\partial \lambda}\right)\right] =$$

$$= \frac{1}{na^2\eta \sin i}\left[\frac{\partial R}{\partial \Omega} + 2\sin^2 i/2 \left(\frac{\partial R}{\partial \tilde{\omega}} + \frac{\partial R}{\partial \lambda}\right)\right]$$

$$\dot{\Omega} = -\frac{1}{na^2\eta \sin i}\frac{\partial R}{\partial i}$$

$$\dot{\tilde{\omega}} = \dot{\omega} + \dot{\Omega} = -\frac{1}{na^2\eta \sin i}\frac{\partial R}{\partial i} + \frac{\cos i}{na^2\eta \sin i}\frac{\partial R}{\partial i} - \frac{\eta}{na^2 e}\frac{\partial R}{\partial e}$$

$$= -\frac{1}{na^2\eta}\left[\frac{\eta^2}{e}\frac{\partial R}{\partial e} + \frac{\overset{2\sin^2 i/2}{1-\cos i}}{\underset{2\sin i/2 \cos i/2}{\sin i}}\frac{\partial R}{\partial i}\right] = -\frac{1}{na^2\eta}\left[\frac{\eta^2}{e}\frac{\partial R}{\partial e} + \frac{\sin i/2}{\cos i/2}\frac{\partial R}{\partial i}\right]$$

$$\dot{\lambda} = \dot{\Omega} + \dot{\omega} + \dot{l} = n + \frac{2}{na}\frac{\partial R}{\partial a} + \frac{\eta^2}{na^2 e}\frac{\partial R}{\partial e} - \frac{\eta}{na^2 e}\frac{\partial R}{\partial e} +$$

$$+ \frac{\cos i}{na^2\eta \sin i}\frac{\partial R}{\partial i} - \frac{1}{na^2\eta \sin i}\frac{\partial R}{\partial i} =$$

$$= n + \frac{2}{na}\frac{\partial R}{\partial a} - \frac{\eta}{na^2 e}(1-\eta)\frac{\partial R}{\partial e} - \frac{1-\cos i}{na^2\eta \sin i}\frac{\partial R}{\partial i}$$

$$= n + \frac{2}{na}\frac{\partial R}{\partial a} - \frac{1}{na^2\eta}\left[\frac{e\eta^2}{1+\eta}\frac{\partial R}{\partial e} + \frac{\sin i/2}{\cos i/2}\frac{\partial R}{\partial i}\right]$$

Nyní definujeme dvojice nesingulárních elementů

$$(e, \tilde{\omega}) \longrightarrow (k, h) \longrightarrow z$$

$$(i, \Omega) \longrightarrow (q, p) \longrightarrow \zeta$$

$$z = e\exp(i\tilde{\omega}) = e\cos\tilde{\omega} + i e\sin\tilde{\omega} = k + ih$$

$$\zeta = \sin i/2 \exp(i\Omega) =$$

$$= \sin i/2 \cos\Omega + i \sin i/2 \sin\Omega = q + ip$$

$$k = e\cos\varpi \qquad e = \sqrt{k^2+h^2} = \sqrt{z\bar{z}}$$
$$h = e\sin\varpi \qquad \operatorname{tg}\varpi = h/k$$

nyní budeme potřebovat

$$\frac{\partial}{\partial e} = \frac{1}{e}\left[k\frac{\partial}{\partial k} + h\frac{\partial}{\partial h}\right], \quad \frac{\partial}{\partial \varpi} = -h\frac{\partial}{\partial k} + k\frac{\partial}{\partial h}$$

a též $\quad k = \frac{1}{2}(z+\bar{z}), \quad h = \frac{i}{2}(\bar{z}-z)$

$$\frac{\partial}{\partial z} = \frac{\partial k}{\partial z}\frac{\partial}{\partial k} + \frac{\partial h}{\partial z}\frac{\partial}{\partial h} = \frac{1}{2}\left[\frac{\partial}{\partial k} - i\frac{\partial}{\partial h}\right]$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial k}{\partial \bar{z}}\frac{\partial}{\partial k} + \frac{\partial h}{\partial \bar{z}}\frac{\partial}{\partial h} = \frac{1}{2}\left[\frac{\partial}{\partial k} + i\frac{\partial}{\partial h}\right], \quad \text{a tudíž}$$

$$\frac{\partial}{\partial k} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \; ; \quad \frac{\partial}{\partial h} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right) \quad \text{a konečně}$$

$$\frac{\partial}{\partial e} = \frac{1}{e}\left[z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}\right] \; ; \quad \frac{\partial}{\partial \varpi} = i\left[z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\right]$$

Implementujeme tedy změnu $\quad (e,\varpi) \mapsto (k,h) \mapsto z$

$$\dot{k} = \dot{e}\cos\varpi - e\sin\varpi\,\dot{\varpi}$$
$$\dot{h} = \dot{e}\sin\varpi + e\cos\varpi\,\dot{\varpi} \qquad \text{tudíž}$$

$$\dot{z} = \dot{k} + i\dot{h} = \frac{\dot{e}}{e}z + \dot{\varpi}\,iz = z\left[\frac{\dot{e}}{e} + i\dot{\varpi}\right]$$

tedy pokračováním a dosazením z Lagrangeových rovnic

$$\dot{z} = z\left[\frac{\eta}{na^2e^2}\frac{\partial R}{\partial \varpi} + \frac{\eta}{na^2(1+\eta)}\frac{\partial R}{\partial \lambda} - i\left(\frac{\eta}{na^2e}\frac{\partial R}{\partial e} + \frac{1}{na^2\eta}\frac{\sin i/2}{\cos i/2}\frac{\partial R}{\partial i}\right)\right]$$

$$= \frac{z}{na^2}\left[\frac{\eta}{e^2}\left(\frac{\partial R}{\partial \varpi} - ie\frac{\partial R}{\partial e}\right) - \frac{i}{\eta}\frac{\sin i/2}{\cos i/2}\frac{\partial R}{\partial i} + \frac{\eta}{1+\eta}\frac{\partial R}{\partial \lambda}\right]$$

Nyní se snadno přesvědčíme, že

$$\frac{\partial}{\partial \varpi} + ie\frac{\partial}{\partial e} = \dots = -2i\bar z\frac{\partial}{\partial \bar z}$$

a tedy

$$\boxed{\dot z = \frac{1}{na^2}\left[\frac{\eta z}{e^2}(-2i\bar z)\frac{\partial R}{\partial \bar z} - \frac{z i}{\eta}\frac{\sin i/2}{\cos i/2}\frac{\partial R}{\partial i} + \frac{\eta}{1+\eta}z\frac{\partial R}{\partial \lambda}\right]}$$

$$= -\frac{1}{na^2}\left[2i\eta\frac{\partial R}{\partial \bar z} + \frac{z i}{\eta}\frac{\sin i/2}{\cos i/2}\frac{\partial R}{\partial i} - \frac{\eta}{1+\eta}z\frac{\partial R}{\partial \lambda}\right]$$

$$\boxed{= -\frac{i}{na^2}\left[2\eta\frac{\partial R}{\partial \bar z} + \frac{z}{\eta}\frac{\sin i/2}{\cos i/2}\frac{\partial R}{\partial i} + \frac{\eta}{1+\eta}iz\frac{\partial R}{\partial \lambda}\right]}$$

Nyní podobné výrazy pro substituci $(i,\Omega)\to(q,p)\to\zeta$

$$q = \sin i/2\cos\Omega \qquad \sin i/2 = \sqrt{q^2+p^2} = \sqrt{\zeta\bar\zeta}$$
$$p = \sin i/2\sin\Omega \qquad \cos i/2 = \sqrt{1-\zeta\bar\zeta}$$

potřebné výrazy: $q = \frac{1}{2}(\zeta+\bar\zeta)$, $p = \frac{i}{2}(\bar\zeta-\zeta)$

$$\frac{\partial}{\partial i} = \frac{1}{2}\frac{\cos i/2}{\sin i/2}\left[q\frac{\partial}{\partial q} + p\frac{\partial}{\partial p}\right]$$

$$\frac{\partial}{\partial \Omega} = -p\frac{\partial}{\partial q} + q\frac{\partial}{\partial p}$$

a protože

$$\frac{\partial}{\partial \zeta} = \frac{1}{2}\left[\frac{\partial}{\partial q} - i\frac{\partial}{\partial p}\right], \quad \frac{\partial}{\partial \bar\zeta} = \frac{1}{2}\left[\frac{\partial}{\partial q} + i\frac{\partial}{\partial p}\right]$$

$$\frac{\partial}{\partial q} = \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \bar\zeta}, \quad \frac{\partial}{\partial p} = i\left(\frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \bar\zeta}\right)$$

máme konečně

$$\frac{\partial}{\partial i} = \frac{1}{2}\frac{\cos i/2}{\sin i/2}\left[\zeta\frac{\partial}{\partial \zeta} + \bar\zeta\frac{\partial}{\partial \bar\zeta}\right] \quad \text{a} \quad \frac{\partial}{\partial \Omega} = i\left[\zeta\frac{\partial}{\partial \zeta} - \bar\zeta\frac{\partial}{\partial \bar\zeta}\right]$$

Můžeme tedy patřičnou derivaci $\partial/\partial i$ doplnit do rovnice pro $\dot{z}$:

$$\boxed{\dot{z} = -\frac{i}{na^2}\left[2\eta\frac{\partial R}{\partial \bar{z}} + \frac{z}{2\eta}\left(\xi\frac{\partial R}{\partial \xi} + \bar{\xi}\frac{\partial R}{\partial \bar{\xi}}\right) + \frac{\eta}{1+\eta}\, i z\frac{\partial R}{\partial \lambda}\right]}$$

Nyní přistoupíme k modifikaci rovnic

$$\dot{q} = \frac{1}{2}\frac{\cos i/2}{\sin i/2}\, q\dot{I} - p\dot{\Omega}$$

$$\dot{p} = \frac{1}{2}\frac{\cos i/2}{\sin i/2}\, p\dot{I} + q\dot{\Omega} \qquad \text{tedy}$$

$$\dot{\xi} = \dot{q} + i\dot{p} = \dots = \xi\left[\frac{1}{2}\frac{\cos i/2}{\sin i/2}\dot{I} + i\dot{\Omega}\right]$$

a dosazením z Lagrangeových rovnic

$$\dot{\xi} = \frac{\xi}{na^2\eta\sin I}\left[\frac{1}{2}\frac{\cos I/2}{\sin I/2}\left(\frac{\partial R}{\partial \Omega} + 2\sin^2 I/2\left(\frac{\partial R}{\partial \varpi} + \frac{\partial R}{\partial \lambda}\right)\right) - i\frac{\partial R}{\partial I}\right] =$$

$$= \frac{\xi}{na^2\eta\sin I}\left[\left(\frac{1}{2}\frac{\cos i/2}{\sin i/2}\frac{\partial}{\partial \Omega} - i\frac{\partial}{\partial I}\right)R + \frac{1}{2}\sin I\left(\frac{\partial R}{\partial \varpi} + \frac{\partial R}{\partial \lambda}\right)\right]$$

nyní $\frac{1}{2}\frac{\cos i/2}{\sin i/2}\frac{\partial}{\partial \Omega} - i\frac{\partial}{\partial I} = \dots = -i\frac{\cos I/2}{\sin I/2}\,\bar{\xi}\frac{\partial}{\partial \bar{\xi}}$ a tudíž

$$\dot{\xi} = \frac{1}{na^2\eta}\left\{-\frac{i\xi\bar{\xi}}{\sin I}\frac{\cos I/2}{\sin I/2}\frac{\partial R}{\partial \bar{\xi}} + \frac{1}{2}\xi\left(\frac{\partial R}{\partial \varpi} + \frac{\partial R}{\partial \lambda}\right)\right\}$$

$$= -\frac{1}{2na^2\eta}\left[i\frac{\partial R}{\partial \bar{\xi}} - \xi\left(\frac{\partial R}{\partial \varpi} + \frac{\partial R}{\partial \lambda}\right)\right] =$$

a dosazením za $\partial R/\partial \varpi$

$$= -\frac{i}{2na^2\eta}\left[\frac{\partial R}{\partial \bar{\xi}} - \xi\left(z\frac{\partial R}{\partial z} - \bar{z}\frac{\partial R}{\partial \bar{z}}\right) + i\xi\frac{\partial R}{\partial \lambda}\right]$$

Můžeme tedy dát dohromady rovnice v nesingulárních elementech (a, z, ξ, λ):

$(\eta = \sqrt{1 - z\bar{z}})$

$$\boxed{\begin{aligned}
\dot{a} &= -\frac{2}{na}\frac{\partial R}{\partial \lambda} \\
\dot{z} &= -\frac{i}{na^2}\left[2\eta\frac{\partial R}{\partial \bar{z}} + \frac{z}{2\eta}\left(\xi\frac{\partial R}{\partial \xi} + \bar{\xi}\frac{\partial R}{\partial \bar{\xi}}\right) + \frac{\eta}{1+\eta}\, i z\frac{\partial R}{\partial \lambda}\right] \\
\dot{\xi} &= -\frac{i}{2na^2\eta}\left[\frac{\partial R}{\partial \bar{\xi}} - \xi\left(z\frac{\partial R}{\partial z} - \bar{z}\frac{\partial R}{\partial \bar{z}}\right) + i\xi\frac{\partial R}{\partial \lambda}\right] \\
\dot{\lambda} &= n + \frac{2}{na}\frac{\partial R}{\partial a} - \frac{1}{na^2\eta}\left[\frac{\eta^2}{1+\eta}\left(z\frac{\partial R}{\partial z} + \bar{z}\frac{\partial R}{\partial \bar{z}}\right) + \right. \\
&\qquad \left. + \frac{1}{2}\left(\xi\frac{\partial R}{\partial \xi} + \bar{\xi}\frac{\partial R}{\partial \bar{\xi}}\right)\right]
\end{aligned}}$$

Přijmeme-li princip středování $R \neq R(\lambda)$ pak $\dot{a} = 0$ ($\rightarrow a = \text{konst.}$) a aktivní rovnice pro (z, ξ) jsou

$$\boxed{\begin{aligned}
\dot{z} &= -\frac{i}{na^2}\left[2\eta\frac{\partial R}{\partial \bar{z}} + \frac{z}{2\eta}\left(\xi\frac{\partial R}{\partial \xi} + \bar{\xi}\frac{\partial R}{\partial \bar{\xi}}\right)\right] \\
\dot{\xi} &= -\frac{i}{2na^2\eta}\left[\frac{\partial R}{\partial \bar{\xi}} - \xi\left(z\frac{\partial R}{\partial z} - \bar{z}\frac{\partial R}{\partial \bar{z}}\right)\right]
\end{aligned}} \qquad (S)$$

Uveďme nyní teorii týkající se sekulárního systému (S): (7

Nechť $R = \Phi(z\bar{z}, \zeta\bar{\zeta}) = \Phi(A,B)$ je funkce pouze $\underline{e}$ a $\underline{i}$ a ne $\underline{\varpi, \Omega}$. Pak (S) má tvar

$$\dot{z} = -i \cancel{k} \frac{1}{na^2}\left[2\eta \Phi_A + \frac{\sin^2 i/2}{\eta}\Phi_B\right] z$$

$$\dot{\zeta} = -i \frac{1}{2na^2\eta}\Phi_B \zeta$$

pak řešení jsou $z = z_0 e^{i\psi_1 t}$, $\zeta = \zeta_0 e^{i\psi_2 t}$ kde

$$\psi_1 = -\frac{1}{na^2}\left[2\eta \Phi_A + \frac{\sin^2 i/2}{\eta}\Phi_B\right]$$

$$\psi_2 = -\frac{1}{2na^2\eta}\Phi_B$$

jsou frekvence sekulárního stáčení ϖ a Ω.

Snadno se totiž přesvědčíme, že

$$\frac{d}{dt}(z\bar{z}) = z\dot{\bar{z}} + \dot{z}\bar{z} = z(-i\psi_1\bar{z}) + i\psi_1 z\bar{z} = 0$$

a podobně pro $\zeta\bar{\zeta}$; amplituda se tedy zachovává a (z,ζ) cirkulují s frekvencemi ψ_1 a ψ_2.

Příkladem může být J_2-potenciál, kdy

$$R = -\frac{GM}{4a}\left(\frac{R}{a}\right)^2 J_2 \frac{2-3\sin^2 i}{\eta^3} = K\frac{2-3\sin^2 i}{\eta^3}$$

$$= 2K(1-A)^{-3/2}[1-6B(1-B)] \quad \text{kde } A = z\bar{z},\ B = \zeta\bar{\zeta}$$

Přímý výpočet frekvencí ψ_1 a ψ_2 dává

$$\psi_1 = \ldots = -\frac{3}{4} n \left(\frac{R}{a}\right)^2 J_2 \, \frac{1+2\cos i - 5\cos^2 i}{\eta^4} = \dot{\omega}$$

$$\psi_2 = \ldots = -\frac{3}{2} n \left(\frac{R}{a}\right)^2 J_2 \, \frac{\cos i}{\eta^4} = \dot{\Omega}$$

kde jsme využili $n^2 a^3 = GM$.

také můžeme určit $\dot{\omega} = \dot{w} - \dot{\Omega} = -\frac{3}{4} n \left(\frac{R}{a}\right)^2 J_2 \, \frac{1-5\cos^2 i}{\eta^4}$

dráhy s $\dot{\omega} = 0 \rightarrow \boxed{\cos i = \pm 1/\sqrt{5}}$ $i \simeq 63°$, $117°$

dráhy s $\dot{w} = 0 \rightarrow$ $i \simeq 47°$, $\simeq 107°$

dráhy s $\dot{\Omega} = 0 \rightarrow \boxed{i = 90°}$

Pro mnohé problémy je též výhodné zavést jinou sadu nesingulárních elementů

$$(a, z, I, \Omega, \ell) \qquad z = e\exp(i\omega)$$

to nastává zvláště v případě poruch R s axiální symetrií, kdy $R \neq R(\Omega)$. Pokud připomeneme původní sadu Lagrangeových rovnic

$$\boxed{\begin{aligned}
\dot a &= -\frac{2}{na}\frac{\partial R}{\partial \ell} \\
\dot e &= -\frac{\eta}{na^2 e}\left[\eta\frac{\partial R}{\partial \ell} - \frac{\partial R}{\partial \omega}\right] \\
\dot I &= -\frac{1}{na^2\eta\sin I}\left[\cos I\frac{\partial R}{\partial \omega} - \frac{\partial R}{\partial \Omega}\right] \\
\dot\Omega &= -\frac{1}{na^2\eta\sin I}\frac{\partial R}{\partial I} \\
\dot\omega &= -\frac{1}{na^2\eta}\left[\frac{\eta^2}{e}\frac{\partial R}{\partial e} - \frac{\cos I}{\sin I}\frac{\partial R}{\partial I}\right] \\
\dot\ell &= n + \frac{2}{na}\frac{\partial R}{\partial a} + \frac{\eta^2}{na^2 e}\frac{\partial R}{\partial e}
\end{aligned}}$$

is now $R \neq R(\Omega, \ell)$ (axiální symetrie + středování)

pak

$$\boxed{\begin{aligned}
e\dot e &= +\frac{\eta}{na^2}\frac{\partial R}{\partial \omega} \\
\sin I\,\dot I &= -\frac{\cos I}{na^2\eta}\frac{\partial R}{\partial \omega}
\end{aligned}}$$

což se kombinuje do
$$\boxed{\frac{e\dot e}{\eta} + \frac{\sin I\,\dot I}{\cos I}\eta = 0}$$

a vede na
$$\boxed{\frac{d}{dt}\left[\cos I\sqrt{1-e^2}\right] = 0}$$

to je vlastně vyjádřením zachování projekce momentu hybnosti na v tomto případě se tedy dynamika redukuje na z-vektor, neboť

- I z integrálu $\boxed{\cos I\sqrt{1-e^2}=K}$
- Ω integrací $d\Omega/dt = \dots$
- $a = \mathrm{konst}$

Najdeme tedy $z = k+ih = e\cos\omega + ie\sin\omega$

Podle str. 3

$$e\frac{\partial}{\partial e} = z\frac{\partial}{\partial z} + \bar z\frac{\partial}{\partial \bar z} \quad ; \quad \frac{\partial}{\partial \omega} = i\left[z\frac{\partial}{\partial z} - \bar z\frac{\partial}{\partial \bar z}\right]$$

$$\boxed{\dot z = z\left[\frac{\dot e}{e} + i\dot\omega\right]} =$$

$$= +\frac{z}{na^2}\left[\frac{\eta}{e^2}\frac{\partial R}{\partial \omega} - i\frac{\eta}{e}\frac{\partial R}{\partial e} + i\frac{\cos I}{\eta\sin I}\frac{\partial R}{\partial I}\right]$$

$$= \frac{z}{na^2}\left[\frac{\eta}{e^2}\left(\frac{\partial R}{\partial \omega} - ie\frac{\partial R}{\partial e}\right) + \frac{\cos I}{\eta\sin I}\frac{\partial R}{\partial I}\right] =$$

$$= \frac{z}{na^2}\left[-2i\frac{\eta}{e^2}\bar z\frac{\partial R}{\partial \bar z} + i\frac{\cos I}{\eta\sin I}\frac{\partial R}{\partial I}\right] =$$

$$\boxed{= -\frac{i}{na^2}\left[2\eta\frac{\partial R}{\partial \bar z} - \frac{z}{\eta}\frac{\cos I}{\sin I}\frac{\partial R}{\partial I}\right]}$$

celá dynamika je obsažena v této rovnici $z(t)$ neboť $I = I(K,\eta)$ dle ↑. To je ještě velmi složitá rovnice, ale v případě (J_2, J_3) členů se omezíme na řešení pro malé e a pak se věc zjednoduší a úloha bude řešitelná.

V případě J_2 potenciálu je to ještě jednodušší a zna

$$\boxed{\dot{z} = \frac{3i}{na^2}K\frac{1-5\cos^2 i}{\eta^4}z}$$ kde $K = -\frac{GM}{4a}\left(\frac{R}{a}\right)^2 J_2$

úhel ω tedy cirkuluje s konstantní frekvencí

$$\boxed{\dot{\omega} = \frac{3K}{na^2}\frac{1-5\cos^2 i}{\eta^4} = -\frac{3}{4}n\left(\frac{R}{a}\right)^2 J_2 \frac{1-5\cos^2 i}{\eta^4}}$$

Pro obecný případ, kdy $R \neq R(\Omega, \ell)$ máme ještě k dispozici následující integrál: $R = R(e, I, \omega; a) = K$

Lagrangeovy rovnice mají tvar $(a = \text{konst.})$.

$$\dot{e} = +\frac{\eta}{na^2 e}\left(\frac{\partial R}{\partial \omega}\right)$$

$$\dot{I} = -\frac{1}{na^2\eta}\frac{\cos I}{\sin I}\left(\frac{\partial R}{\partial \omega}\right)$$

$$\dot{\omega} = -\frac{\eta}{na^2 e}\left(\frac{\partial R}{\partial e}\right) + \frac{1}{na^2\eta}\frac{\cos I}{\sin I}\left(\frac{\partial R}{\partial I}\right)$$

Thus

$$\boxed{\frac{d}{dt}R(e, I, \omega) = \frac{\partial R}{\partial e}\dot{e} + \frac{\partial R}{\partial I}\dot{I} + \frac{\partial R}{\partial \omega}\dot{\omega} = \dots = 0}$$

Toto ostatně platí i pro obecný případ kdy jen $R \neq R(\ell)$

I pak $R(e, I, \Omega, \omega; a) = K$.

Vliv J_3 člen: Deprit & Coffey dávají

check

$$R = \frac{\mu}{4a}\left(\frac{R}{a}\right)^2 J_2 \frac{1-3\cos^2 I}{\eta^3} + \frac{3\mu}{8a}\left(\frac{R}{a}\right)^3 J_3 \frac{e\sin\omega}{\eta^5}\sin I\,(1-5\cos^2 I)$$

$$= \frac{\mu}{4a}\left(\frac{R}{a}\right)^2 J_2 \frac{1-3\cos^2 I}{\eta^3} + \frac{3\mu}{16a}\left(\frac{R}{a}\right)^3 J_3 \frac{i(\bar{z}-z)}{\eta^5}\sin I\,(1-5\cos^2 I)$$

dosadíme-li toto $\underline{R}$ do rovnice pro $\underline{z}$-dynamiku a omezíme se na malé okolí počátku (zanedbáme členy $\simeq z\bar{z} \simeq e^2$), dostáváme

$$\boxed{\dot{z} = i\Omega_2 z + T_3}\ (+O(e^2))$$

v této aproximaci též $I \simeq$ konst.

$$\Omega_2 = -\frac{3}{4}n\left(\frac{R}{a}\right)^2 J_2\,(1-5\cos^2 I)$$

$$T_3 = \frac{3}{8}n\left(\frac{R}{a}\right)^3 J_3 \sin I\,(1-5\cos^2 I)$$

Výše napsané rovnice je soustava lineárních dif. rovnic s pravou stranou T_3; partikulární řešení je

$$i\Omega_2 z_p + T_3 = 0 \quad , \text{tj.}$$

$$\boxed{z_p = iT_3/\Omega_2 = -i\,\frac{1}{2}\left(\frac{R}{a}\right)\frac{J_3}{J_2}\sin I}$$

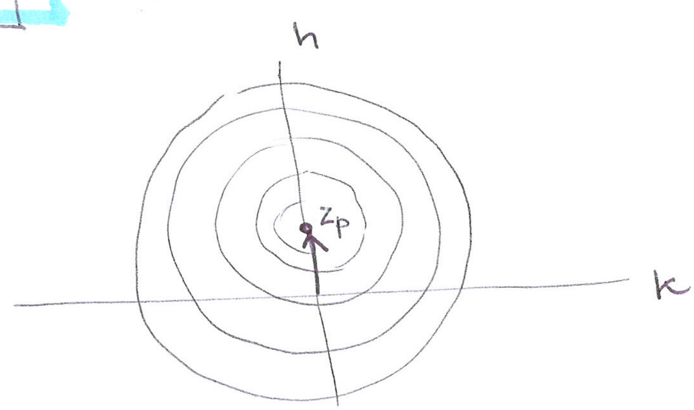
$J_2 = 0.00108263..$

$J_3 = -2.532.. \times 10^{-6}$

$$e_p = -\frac{1}{2}\left(\frac{R}{a}\right)\frac{J_3}{a}\sin I \approx -\frac{1}{2}\frac{J_3}{J_2} \simeq$$

$$\simeq 0.00234..$$

a málo mění pro I blízké $0°$.



obecně portrét křivek v (k,h) rovině plyne z $\underline{R = konst.}$, ale pak má smysl započíst i další jevy. Pro dráhu u míchy je $\underline{e}$ blízké $\underline{e}_p$ úhel $\underline{\omega}$ osciluje kolem $90^0 \rightarrow$ př.