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Unified analytical solutions to two-body problems with drag

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ABSTRACT
The two-body problem with a generalized Stokes drag is discussed. The drag force is proportional to the product of the velocity vector and the inverse square of the distance. The generalization consists of allowing two different proportionality constants for the radial and the transverse components of the force. Under the 'generalized Robertson transformation', the equation of the orbit takes the form of the Lommel equation and admits solutions in terms of Bessel and Lommel functions. The exact, analytical solutions for this type of drag reveal a paradoxical effect of increasing eccentricity for all trajectories. The Poynting–Robertson drag and Poynting–Plummer–Danby problems are discussed as particular cases of the general solution.

Key words: celestial mechanics, stellar dynamics.

1 INTRODUCTION

In the excellent reviews (Burns, Lamy & Soter 1979; Mignard 1992) of the orbital motion of particles under the influence of radiation forces, extensive attention is given to the perturbation solutions of the problem. Wyatt & Whipple (1950) draw on perturbation treatment in the original paper by Robertson (1937) to derive a 'quasi-integral' in the case of Poynting–Robertson (PR) drag. It is interesting, however, that the exact solution found by Robertson (1937) for PR drag seems to have been forgotten. Even Robertson himself does not mention his reduction of the problem to an inhomogeneous Bessel equation in his later book (Robertson & Noonan 1968). The practical aspects of the first-order perturbation theory are quite important, but an analytic solution to the PR problem in those astrophysical situations where a small parameter expansion is superseded could be quite useful.

Further, by framing the problem in a non-perturbation formulation one may extend the context to a more unified treatment of problems of orbits with dissipative forces. Robertson (1937) noted in his paper the similar forms of the equations of orbital drag motion derived by Poynting (1903) and Plummer (1905). Plummer even developed and studied a second model of radiation effects (Plummer 1906). These researches appear as an example in Plummer's monograph (1918) and then in Danby's 'Fundamentals of Celestial Mechanics' (1962). The problem acquired new life in the papers of Mittleman & Jezewski (1982) and Mavraganis & Michalakis (1994). This 'Poynting–Plummer–Danby' drag can be interpreted as a Stokes drag in a viscous medium surrounding a central body, where the resistance of the medium is inversely proportional to the square of the distance from the central body, and taking the moving body to have the drag characteristics of a spherical object. The Stokes drag, although improperly applied to the radiation pressure problem by Poynting (1903) and Plummer (1905), frequently occurs in cosmogonic and astrophysical problems [cf. Ferraz-Mello (1992) and references therein].

The aim of this paper is to return to Robertson's first approach, and to generalize it such that the Poynting–Robertson problem and the Poynting–Plummer problems can be treated as particular cases of a general solution.

2 GENERALIZED ROBERTSON TRANSFORMATION

Suppose that a particle moving around a primary point mass m_0 is subject to a perturbing force

$$\boldsymbol{P} = -r^{-2}\,(\alpha_\mathrm{r}\,\boldsymbol{v}_\mathrm{r} + \alpha_\mathrm{t}\,\boldsymbol{v}_\mathrm{t}), \tag{1}$$

where r is the distance between the masses, $\boldsymbol{v}_\mathrm{r}$ is the radial velocity, $\boldsymbol{v}_\mathrm{t}$ is the transverse velocity, and α_r, α_t are some real, positive constants. In the reference frame with the centre at m_0, the equations of motion for polar variables r, ϑ are

$$\begin{cases} \ddot r - r\dot\vartheta^2 &= -\mu\, r^{-2} - \alpha_\mathrm{r}\,\dot r\, r^{-2}, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}(r^2\dot\vartheta) &= -\,\alpha_\mathrm{t}\,\dot\vartheta. \end{cases} \tag{2}$$

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It is worth noting that the constant μ can be different from $k^2\, m_0$ (as happens in the PR problem), including the effect of all radial forces with a potential $V \propto r^{-1}$. We can point out three particular cases of equations (2):

(i) Poynting–Plummer–Danby drag, when $\alpha_\mathrm{r} = \alpha_\mathrm{t}$ (Poynting 1903; Plummer 1905; Danby 1962);
(ii) Poynting–Robertson drag, when $\alpha_\mathrm{r} = 2\,\alpha_\mathrm{t}$ (Robertson 1937);
(iii) Poynting–Plummer drag, when $\alpha_\mathrm{r} = 4\,\alpha_\mathrm{t}$ (Plummer 1906).

Equations (2) admit a first integral for angular momentum H (Poynting 1903):

$$H = H_0 - \alpha_\mathrm{t}\,(\vartheta - \vartheta_0) = h - \alpha_\mathrm{t}\,\vartheta, \tag{3}$$

where H_0 and ϑ_0 are the values of the angular momentum and of the true longitude at some initial epoch t_0. Following Poynting's treatment, which is essentially the Binet transformation $(r(t), \vartheta(t)) \longrightarrow (u(\vartheta), t(\vartheta))$ with $u = r^{-1}$, but modified by the variability of H, we would obtain equations (2) as

$$\begin{cases} \dfrac{\mathrm{d}^2 u}{\mathrm{d}\vartheta^2} + \dfrac{\alpha_\mathrm{r} - \alpha_\mathrm{t}}{h - \alpha_\mathrm{t}\,\vartheta}\,\dfrac{\mathrm{d}u}{\mathrm{d}\vartheta} + u & = \dfrac{\mu}{(h - \alpha_\mathrm{t}\,\vartheta)^2}, \\ \dfrac{\mathrm{d}t}{\mathrm{d}\vartheta} & = \dfrac{1}{u^2 (h - \alpha_\mathrm{t}\vartheta)}. \end{cases} \tag{4}$$

Following Robertson (1937), we can go further, applying the transformation $(r(t), \vartheta(t)) \longrightarrow (y(x), t(x))$, with

$$x = H\,\alpha_\mathrm{t}^{-1} = h\,\alpha_\mathrm{t}^{-1} - \vartheta,$$

$$y = \frac{\alpha_\mathrm{t}^2}{\mu\, r\, x^\nu}. \tag{5}$$

Our generalization consists of adding a free parameter ν at the place where Robertson set $\nu = 1$. Under this 'generalized Robertson transformation', equations (2) are replaced by

$$\begin{cases} x^2\, y'' + (2\,\nu + 1 - \alpha_\mathrm{r}/\alpha_\mathrm{t})\, x\, y' + \left[x^2 + \nu\left(\nu - \alpha_\mathrm{r}/\alpha_\mathrm{t}\right)\right] y & = x^{-\nu}, \\ t' + \alpha_\mathrm{t}^3\, \mu^{-2}\, x^{-2\nu - 1}\, y^{-2} & = 0, \end{cases} \tag{6}$$

where the prime stands for the derivative with respect to x. Now, we can choose

$$\nu = \frac{\alpha_\mathrm{r}}{2\,\alpha_\mathrm{t}}, \tag{7}$$

and the first of equations (6) – the equation of orbit – becomes

$$x^2\, y'' + x\, y' + (x^2 - \nu^2)\, y = x^{-\nu}. \tag{8}$$

This inhomogeneous Bessel equation of order ν is the particular case of a Lommel equation.

3 LOMMEL FUNCTIONS AND ARBITRARY CONSTANTS

For a detailed discussion of equation (8), the reader is referred to Watson's *magnum opus* on Bessel functions (Watson 1958). The general solution has the form

$$y(x) = Z_\nu(x) + S_{-(\nu+1),\nu}(x), \tag{9}$$

where $S_{-(\nu+1),\nu}$ is a Lommel function, and

$$Z_\nu(x) = A\, J_\nu(x) + B\, Y_\nu(x), \tag{10}$$

where A, B are arbitrary constants accompanying the Bessel functions of the first and the second kind J_ν, Y_ν. For the sake of brevity, we designate the subset of Lommel functions $S_{\kappa,\nu}$ with $\kappa = -(\nu + 1)$ as $\hat{S}_\nu$, and ocasionally we omit the arguments of functions.

All functions $\hat{S}_\nu$ can be represented as the asymptotic series for $x \gg 1$:

$$\hat{S}_\nu(x) \sim \frac{1}{x^{\nu+2}\, \Gamma(\nu + 1)} \sum_{m \geq 0} (-4)^m\, x^{-2m}\, m!\, \Gamma(\nu + m + 1). \tag{11}$$

Note that the series is divergent, but for large values of x it assures a sufficiently good accuracy with just a couple of leading terms. For small values of x, the situation is a bit more involved. For non-integer values of ν the series

$$\hat{S}_\nu(x) = \frac{\Gamma(-\nu)}{4} \left[\sum_{m=0}^{\infty} \frac{(-1)^m\, x^{2m-\nu}\, 2^{-2m}}{\Gamma(m - \nu + 1)\, m!} \left[2 \ln \frac{x}{2} - \psi(m - \nu + 1) - \psi(m + 1) \right] - \frac{\pi}{2^\nu}\, Y_{-\nu}(x) \right] \tag{12}$$

can be applied. When ν is an integer, the recurrence relation

$$\hat{S}_{\nu+1}(x) = \frac{1}{2\,\nu + 2} \left[\frac{\nu}{x}\, \hat{S}_\nu(x) - \hat{S}'_\nu(x) \right] \tag{13}$$

can be used, taking

$$\hat{S}_0(x) = \frac{1}{2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2}\left(\frac{x}{2}\right)^{2m}\left\{\left[\ln\frac{x}{2} - \psi(m+1)\right]^2 - \frac{\psi'(m+1)}{2} + \frac{\pi^2}{4}\right\} \tag{14}$$

as the starting point. Throughout the paper, Γ, ψ and ψ' stand for the gamma, digamma and trigamma functions respectively (see Abramowitz & Stegun 1965).

To simplify discussion and to obtain hints about more general conclusions we can restrict ourselves to the leading terms of asymptotic expansions, assuming

$$\begin{cases} \hat{S}_\nu & \approx & x^{-\nu-2}, \\ \hat{S}'_\nu & \approx & -(\nu+2)\,x^{-\nu-3}, \\ J_\nu & \approx & \sqrt{\dfrac{2}{\pi x}}\cos\left[(x-(2\nu+1)\pi/4\right], \\ Y_\nu & \approx & \sqrt{\dfrac{2}{\pi x}}\sin\left[x-(2\nu+1)\pi/4\right]. \end{cases} \tag{15}$$

It is worth noting that, when α_t/h is a small parameter, the above approximations are more or less compatible with a first-order approximation.

The determination of arbitrary constants from initial conditions is a straightforward affair. Given ϑ_0, h, r_0 and $\dot{r}_0$ at the initial epoch, one can evaluate x_0, y_0 and y'_0. Then, one readily obtains

$$\begin{cases} A & = & \dfrac{\pi x_0}{2}\left\{[y_0 - \hat{S}_\nu(x_0)]\,Y'_\nu(x_0) - [y'_0 - \hat{S}'_\nu(x_0)]\,Y_\nu(x_0)\right\}, \\ B & = & \dfrac{\pi x_0}{2}\left\{[y'_0 - \hat{S}'_\nu(x_0)]\,J_\nu(x_0) - [y_0 - \hat{S}_\nu(x_0)]\,J'_\nu(x_0)\right\}. \end{cases} \tag{16}$$

4 OSCULATING ELEMENTS

By the definition of the transformation (5) we have

$$r = \frac{\alpha_t^2}{\mu\, x^\nu\,(Z_\nu + \hat{S}_\nu)}, \tag{17}$$

$$\dot{r} = \frac{\mu}{\alpha_t\, x^{\nu-2}}(\nu\, y + x\, y') = \frac{\mu}{\alpha_t\, x^{\nu-2}}(x\, Z_{\nu-1} + \hat{S}_\nu + x\,\hat{S}'_\nu). \tag{18}$$

This information is sufficient to recover the shape and the orientation of the orbit in terms of osculating elements.

The immediate and generally valid consequence of the integral (3) is that the semilatus rectum p is a quadratic function of x (Robertson 1937):

$$p = \frac{H^2}{\mu} = \frac{\alpha^2\, x^2}{\mu}. \tag{19}$$

This is a good place to make the remark that, owing to the definition (5), our variable x decreases when ϑ grows, and that dealing with drag one should beware the habit of treating the true longitude modulo $2\,\pi$.

To determine the eccentricity e and the true anomaly f we turn to

$$\begin{cases} e\,\cos f & = & \dfrac{p}{r} - 1 & = & x^{\nu+2}(Z_\nu + \hat{S}_\nu) - 1, \\ e\,\sin f & = & \sqrt{\dfrac{p}{\mu}}\,\dot{r} & = & x^{\nu+2}\left(Z_{\nu-1} + \dfrac{\nu}{x}\,\hat{S}_\nu + \hat{S}'_\nu\right). \end{cases} \tag{20}$$

The longitude of pericentre is obtained as

$$\varpi = \vartheta - f \; = \; h\,\alpha_t^{-1} - x - f. \tag{21}$$

Substituting approximations (15) into equations (20) we have

$$e^2 \approx \frac{2}{\pi}\, x^{2\nu+3}(A^2+B^2) + \frac{4}{x^2} - \frac{4}{\sqrt{\pi}}\, x^{\nu+1/2}[(A+B)\,\cos(x-\nu\pi/2) - (A-B)\,\sin(x-\nu\pi/2)]. \tag{22}$$

Four interesting conclusions emerge from this crude approximation.

(i) When α_t/h is a small quantity and x is as large as h/α_t, we have

$$e \approx \sqrt{2/\pi}\; x^{\nu+3/2}\,\sqrt{A^2+B^2}, \tag{23}$$

and we may conclude that the arbitrary constants A, B are of the order of $(\alpha_t/h)^{\nu+3/2}$.

(ii) Under the same conditions, e has no terms periodic in x (hence in ϑ) in the first approximation with respect to α_t/h.

(iii) The 'pseudo-integral' derived by Wyatt & Whipple (1950) from the first-order averaged equations of Robertson (1937),

$$p\,e^{-4/5} = C, \tag{24}$$

can be generalized. Imposing the condition that the exponent of x in the product

$$p\,e^{m} \approx \frac{\alpha_t^2\,x^2}{\mu}\left(\sqrt{2/\pi}\,x^{\nu+3/2}\,\sqrt{A^2+B^2}\right)^{m} \tag{25}$$

should be equal 0, we obtain

$$p\,e^{-4/(3+2\nu)} = C \approx \frac{\alpha_t^2}{\mu}\left[\frac{\pi}{2\,(A^2+B^2)}\right]^{2/(3+2\nu)}. \tag{26}$$

Let us add that C is a constant only in the first approximation with respect to α_t/h, but it comes out even without averaging and hence its character is more general than usually presumed. How far is this approximate integral is reliable can be easily checked by means of the rigorously treated equations (19) and (20) in each particular problem considered.

(iv) The presence of the term x^{-2} in equation (22) makes us doubt the common opinion that the osculating eccentricity always decreases systematically under the action of drag forces. We develop this idea in the next sections. Let us note that similar doubts were expressed by Klǎcka & Kaufmannová (1993).

5 PSEUDO-CIRCULAR SOLUTIONS

If $A = B = 0$, there are no terms periodic with respect to ϑ in the particle distance r, and we have $y = \hat{S}_\nu$. Because a spiral-shaped trajectory can hardly be called a circle, let us call the case 'pseudo-circular'. The zero values of our arbitrary constants do not imply that the osculating $e = 0$, but they match the case of the zero mean eccentricity in the sense of Wyatt & Whipple (1950). For the pseudo-circular trajectory equations (20) take the form

$$\begin{cases} e\,\cos f &= x^{\nu+2}\hat{S}_\nu - 1 \approx -4\,(\nu+1)\,x^{-2} + 32\,(\nu+1)\,(\nu+2)\,x^{-4}, \\ e\,\sin f &= x^{\nu+1}\left(\nu\,\hat{S}_\nu + x\,\hat{S}'_\nu\right) \approx -2\,x^{-1} + 16\,(\nu+1)\,x^{-3}, \end{cases} \tag{27}$$

where the first two terms of the asymptotic series (11) have been substituted. Expanding in powers of $1/x$ we obtain for $x \gg 1$

$$e \approx 2\,x^{-1} + 4\,(\nu+1)(\nu-3)\,x^{-3}, \tag{28}$$

$$\tan(f/2) = \frac{e\sin f}{e + e\cos f} \approx -1 - 2\,(\nu+1)x^{-1}. \tag{29}$$

Checking the second extremity (when x tends to 0) by means of equations (12)–(14) for $\hat{S}_\nu$ we find that, whatever the value of $\nu \in \mathbf{R}_+$, the limits are

$$\lim_{x\to 0^+}(e\cos f) = -1,\ \lim_{x\to 0^+}(e\sin f) = 0, \tag{30}$$

and $e\sin f$ approaches zero from the left. We omit the proof, which follows from a more general property discussed in the next section.

The general conclusion is that in the pseudo-circular solution the osculating eccentricity *grows systematically* from the small but non-zero value of order α_t/h towards $e = 1$. The true anomaly remains captured in the interval $\pi < f < 3\pi/2$, slowly regressing towards $f = \pi$ on the time-scale of the entire lifetime of a particle. What makes the particle go around the central body is the fast rotation of the line of apsides. Such unbecoming conduct of the osculating elements in circular cases is nothing new in celestial mechanics. The reader may compare it with the equatorial, circular orbit in the main problem of artificial satellites discussed by Cohen & Lyddane (1981). What is surprising is that the effect passed unnoticed in drag-related problems.

A nice feature of the pseudo-circular trajectories is that they allow us to solve the time-of-flight equation for $x \gg 1$ without much effort. The term $x^{-2\nu-1}y^{-2} = x^{-2\nu-1}\,\hat{S}_\nu^{-2}$ in the second of equations (6) can be expanded asymptotically, and elementary quadratures give us

$$t_1 - t_0 \sim -\alpha_t^3\,\mu^{-2}\left[\frac{1}{4}x^4 + 4\,(\nu+1)\,x^2 - 16\,(\nu^2+6\,\nu+5)\ln x - 128\,(\nu+1)\,(\nu^2+8\,\nu+13)\,x^{-2} + O(x^{-4})\right]_{x_0}^{x_1}. \tag{31}$$

Using this equation one can compute the time interval required by a particle to move from the intial x_0 to the final x_1. Equation (31) permits us to compute $x(t)$ through iterations with a good initial approximation supplied by the first two terms of its right-hand side.

6 TERMINAL VALUES OF OSCULATING ELEMENTS

The pseudo-circular solution is obviously an ideal and very particular case. However, it can be demonstrated that equations (30) are valid for all orbits. Equations (20) can be rewritten as

$$\begin{cases} e\,\cos f &= x^{\nu+2}\,y - 1, \\ e\,\sin f &= \nu\,x^{\nu+1}\,y + x^{\nu+2}\,y'. \end{cases} \tag{32}$$

Hence the sufficient conditions for (30) in the general case are

$$\lim_{x\to 0^+}(x^{\nu+1}\,y) = 0 \quad \text{and} \quad \lim_{x\to 0^+}(x^{\nu+2}\,y') = 0. \tag{33}$$

To simplify the demonstration that equations (33) are satisfied, we can rely on simple estimates:

$$J(x)_\nu \approx \zeta_1\,x^\nu,\, Y(x)_\nu \approx \zeta_2\,x^{-\nu}. \tag{34}$$

Throughout this section, the ζ_i stand for unspecified coefficients having some finite values. Solving equation (8) by means of the variation of arbitrary constants, we obtain (Watson 1958)

$$y = \frac{\pi}{2}\left[Y_\nu \int x^{-\nu-1} J_\nu\,\mathrm{d}x - J_\nu \int x^{-\nu-1}\,Y_\nu\,\mathrm{d}x\right], \tag{35}$$

so that

$$y \approx \zeta_1\,x^{-\nu} + \zeta_2\,x^\nu + \zeta_3\,x^{-\nu}\ln x, \tag{36}$$

and

$$\begin{cases} x^{\nu+1}y & \approx \quad \zeta_1\,x + \zeta_2\,x^{2\nu+1} + \zeta_3\,x\,\ln x, \\ x^{\nu+2}y' & \approx \quad \zeta_1\,x + \zeta_2\,x^{2\nu+1} + \zeta_3\,x^{\nu+2}\ln x + \zeta_4\,x^{\nu+2}. \end{cases} \tag{37}$$

The limits of the right-hand sides of equations (37) for $x \to 0^+$ are 0 and conditions (33) are satisfied, which proves that all trajectories in the problem under discussion terminate at $e = 1, f = \pi$.

7 POYNTING–ROBERTSON PROBLEM

As a particular case of the generalized Robertson transformation let us analyse the Poynting–Robertson effect. Radiation forces act in a twofold manner.

(i) The radial pressure force $k^2 m_0 \beta r^{-2}$ modifies the gravity parameter such that $\mu = k^2 m_0(1-\beta)$. This elementary operation is the only modest triumph of Hoëne-Wroński's (1847) principle, that radial forces can be modelled as an effective gravity parameter instead of being treated as perturbations.

(ii) The velocity-dependent drag matches the model (1) with

$$\alpha_\mathrm{r} = 2\,\alpha_\mathrm{t} = k^2\,m_0\,\beta/c \equiv 2\,\alpha, \tag{38}$$

where c is the speed of light and m_0 is the solar mass. The parameter β depends on the size, density and optical properties of the particle.

Setting our parameter $\nu = 1$, we have the equation of orbit

$$x^2\,y'' + x\,y' + (x^2 - 1)\,y = x^{-1}, \tag{39}$$

with the solution

$$y = A\,J_1 + B\,Y_1 + \hat{S}_1. \tag{40}$$

The asymptotic expansion of $\hat{S}_1$ is exactly the one derived by Robertson (1937):

$$\hat{S}_1 \sim x^{-3} - 8\,x^{-5} + 192\,x^{-7} - 9216\,x^{-9} + \ldots, \tag{41}$$

and it follows from our equation (11). By means of equations (13) and (14) we can obtain the series for small x:

$$\hat{S}_1 = -\frac{\gamma + \ln\frac{x}{2}}{2x} + \frac{x}{8}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,(m+1)!}\left(\frac{x}{2}\right)^{2m}\left\{-1/2\,\psi'(m+2) + \left[\ln\frac{x}{2} - \psi(m+2)\right]^2 + \frac{\ln\frac{x}{2} - \psi(m+2)}{m+1} + \frac{\pi^2}{4}\right\}, \tag{42}$$

where $\gamma = -\psi(1)$ is the Euler gamma constant.

In most realistic applications we can rely on the asymptotic series (41). According to the definition of x it is a positive variable which diminishes during the motion until the terminal value $x = 0$ is reached for all bounded motions. However, taking the radius of the Sun as the lower limit of r, we find that the variable x takes approximately the minimum value

$$x_{\min} \approx 670\,\frac{\sqrt{1-\beta}}{\beta}, \tag{43}$$

and even for β as large as 0.99 it is still large enough to guarantee good precision for asymptotic approximations for Bessel and Lommel functions.

Fig. 1 presents an example of the application of the solution in terms of Lommel functions. As the initial conditions we assumed $e = 0.001$, $a = 2$ au, $f = \varpi = 0$. For a micron-sized particle we took $\beta = 0.2$. The reader may observe that the eccentricity decreases initially but then, after the stage of large periodic variations, it starts to grow. The case presented in Fig. 1 seems to be generic, although not in every case does e start to grow before reaching the solar surface (the time indicated by the dashed line). The growth in eccentricity is associated with the decrease of the pseudo-integral C, defined by equation (24). Let it be recalled that the assumed invariance of C is the cornerstone of the solution of Wyatt & Whipple (1950). In our exemplary case, the reader's confidence in this solution should end after some 9000 revolutions at most.

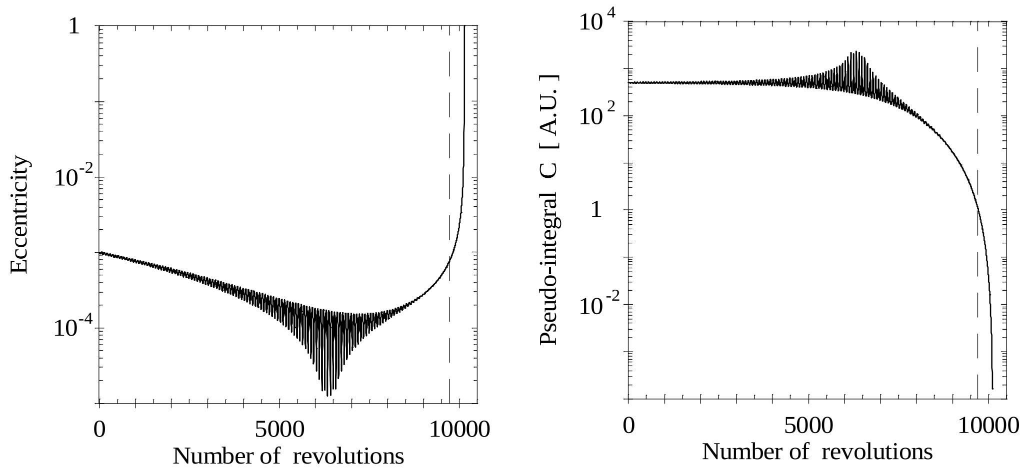

Figure 1. The plots of osculating eccentricity e (left) and $C = p\,e^{-4/5}$ (right) as functions of $\vartheta/2\pi$ in the PR problem. The dashed line indicates the moment of crossing the radius of the Sun.

8 POYNTING–PLUMMER–DANBY PROBLEM

The case of $\alpha_{\rm r} = \alpha_{\rm t} \equiv \alpha$ has the remarkable property that the first of equations (4) becomes a forced harmonic oscillator. Mittleman & Jezewski (1982) achieved progress in what they considered 'Danby's model' by turning back the clock and adopting one part of the Robertson transformation: $x = h/\alpha - \vartheta$. The solution for $u(x)$ that they obtained in terms of sine and cosine integrals can be derived immediately in terms of Lommel functions. With $\nu = 1/2$, the solution of equation (8) is

$$y(x) = A\,J_{1/2}(x) + B\,Y_{1/2}(x) + \hat{S}_{1/2}(x). \tag{44}$$

From equations (5) we have

$$u = \frac{\mu}{\alpha^2}\left(\sqrt{\frac{2}{\pi}}A\,\sin x + \sqrt{\frac{2}{\pi}}B\,\cos x + \sqrt{x}\,\hat{S}_{1/2}\right). \tag{45}$$

Substituting relevant expressions for the Lommel function $\hat{S}_{1/2}$, one can easily check that this result is equivalent to those of Mittleman & Jezewski (1982), or of Mavraganis & Michalakis (1994).

9 CONCLUSIONS

Generalizing the Robertson transformation, we are able to show that there are explicit analytic solutions for a class of two-body problems with drag. We emphasize that the two-body drag model that we have discussed leads to exact solutions of the orbit without any appeal to an expansion in a small parameter. Further, with the ratio ν set to a real, positive number, it should be possible to study the joint action of radiation pressure and Stokes drag. For these classes of drag it is possible to interpret the evolution of a small body in terms of osculating elements without referring to a Gaussian perturbation formulation.

We have established on a firm mathematical basis the fact, occasionally noted by Klačka & Kaufmannová (1993), that the orbital eccentricity decreases only at the initial stage of orbit evolution. Moreover, we have shown that for 'pseudo-circular' orbits the eccentricity can only grow. However, these apparently surprising properties reflect an obvious fact, that all collision orbits should be asymptotically rectilinear (hence with $e = 1, f = \pi$).

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